Finiteness theorems for algebraic groups over function fields

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Abstract

We prove the finiteness of class numbers and Tate-Shafarevich sets for all affine group schemes of finite type over global function fields, as well as the finiteness of Tamagawa numbers and Godement's compactness criterion (and a local analogue) for all such groups that are smooth and connected. This builds on the known cases of solvable and semisimple groups via systematic use of the recently developed structure theory and classification of pseudo-reductive groups.

1. Introduction

1.1 Motivation

The most important classes of smooth connected linear algebraic groups G over a field k are semisimple groups, tori, and unipotent groups. The first two classes are unified by the theory of reductive groups, and if k is perfect then an arbitrary G is canonically built up from all three classes in the sense that there is a (unique) short exact sequence of k-groups

$$1 \to U \to G \to G/U \to 1 \tag{1.1.1}$$

with smooth connected unipotent U and reductive G/U. (Here, U is necessarily a descent of the "geometric" unipotent radical $\mathscr{R}_u(G_{\overline{k}})$ through the *Galois* extension $\overline{k}/k$, and it is k-split.) Consequently, if k is a number field or p-adic field then for many useful finiteness theorems (involving cohomology, volumes, orbit questions, etc.) there is no significant difference between treating the general case and the reductive case.

Over imperfect fields (such as local and global function fields) the unipotent radical $\mathscr{R}_u(G_{\overline{k}})$ in $G_{\overline{k}}$ may not be defined over k (i.e., not descend to a k-subgroup of G). When that happens, G does not admit an extension structure as in (1.1.1). Working with the full radical $\mathscr{R}(G_{\overline{k}})$ is no better; one can make such G that are perfect (i.e., $G = \mathscr{D}(G)$), so $\mathscr{R}(G_{\overline{k}}) = \mathscr{R}_u(G_{\overline{k}})$. Hence, proving a theorem in the solvable and semisimple cases is insufficient to easily deduce an analogous result in general over imperfect fields.

Example 1.1.1. Consider the natural faithful action of $G = \mathrm{PGL}_{nm}$ on $X = \mathrm{Mat}_{nm\times nm}$ with $n, m \geqslant 1$. For a degree-m extension field k'/k admitting a primitive element $a' \in k'^{\times}$, upon choosing an ordered k-basis of k' the resulting element $a' \cdot \mathrm{id}_n \in \mathrm{GL}_n(k') \subseteq \mathrm{GL}_{nm}(k)$ corresponds to a point $x \in X(k)$. The stabilizer G_x of x is isomorphic to the Weil restriction $\mathrm{R}_{k'/k}(\mathrm{PGL}_n)$, so it is smooth and connected. However, this k-group can be bad in two respects.

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Assume k'/k is not separable. The k-group G_x is not reductive [CGP, Ex. 1.1.12, Ex. 1.6.1, Thm. 1.6.2(2),(3)], and no nontrivial smooth connected subgroup of $\mathscr{R}_u((G_x)_{\overline{k}}) = \mathscr{R}((G_x)_{\overline{k}})$ descends to a k-subgroup of G_x [CGP, Prop. 1.1.10, Lemma 1.2.1]. If also $\mathrm{char}(k)|n$ then G_x is not perfect and $\mathscr{D}(G_x) = \mathrm{R}_{k'/k}(\mathrm{SL}_n)/\mathrm{R}_{k'/k}(\mu_n)$ with $\dim \mathrm{R}_{k'/k}(\mu_n) > 0$ [CGP, Prop. 1.3.4, Ex. 1.3.2]. In such cases, by [CGP, Ex. 1.3.5] the k-group $\mathscr{D}(G_x)$ is not isomorphic to $\mathrm{R}_{K/k}(H)/N$ for any finite extension K/k, connected reductive K-group H, and finite normal k-subgroup scheme $N \subset \mathrm{R}_{K/k}(H)$.

The arithmetic of connected semisimple groups over local and global fields rests on the structure theory of semisimple groups over general fields, and this leads to useful finiteness theorems. Examples of such theorems are reviewed in §1.2–§1.3. By separate (typically easier) arguments, these finiteness results often have analogues in the solvable case. Bootstrapping to general G is straightforward when (1.1.1) is available, but over local and global function fields k there are natural questions (e.g., see [CGP, Intro.], which ties in with Example 1.1.1) leading to perfect G not admitting an extension structure over k as in (1.1.1) yet for which one wants analogues of the finiteness theorems that are known in the solvable and semisimple cases.

Despite the general non-existence of (1.1.1) over imperfect fields, the structure theory of pseudo-reductive groups in [CGP] (which was developed due to the needs of this paper) provides support for the following surprising principle (requiring modification in characteristics 2 and 3):

Principle: *To prove a theorem for all smooth connected affine groups over an imperfect field k, it suffices to prove it in the solvable case over k and the semisimple case over finite extensions of k.*

The starting point is a naive-looking generalization of (1.1.1) that makes sense for any smooth connected affine group G over any field k but whose utility is not initially apparent: the short exact sequence

$$1 \to \mathscr{R}_{u,k}(G) \to G \to G/\mathscr{R}_{u,k}(G) \to 1, \tag{1.1.2}$$

where the *k-unipotent radical* $\mathscr{R}_{u,k}(G)$ is the maximal smooth connected unipotent normal k-subgroup of G.

Definition 1.1.2. A k-group G is *pseudo-reductive* if it is smooth, connected, and affine with $\mathscr{R}_{u,k}(G) = 1$.

For any smooth connected affine k-group G, it is clear that the quotient $G/\mathscr{R}_{u,k}(G)$ is pseudo-reductive. Thus, (1.1.2) expresses G (uniquely) as an extension of a pseudo-reductive k-group by a smooth connected unipotent k-group. If k'/k is a separable extension (such as k_s/k, or k_v/k for a place v of a global field k) then $\mathscr{R}_{u,k}(G)_{k'} = \mathscr{R}_{u,k'}(G_{k'})$ inside of $G_{k'}$ [CGP, Prop. 1.1.9(1)]. Hence, if k'/k is separable then G is pseudo-reductive if and only if $G_{k'}$ is pseudo-reductive. If k is perfect then (1.1.2) coincides with (1.1.1) and pseudo-reductivity is the same as reductivity (for connected groups), so the concept offers nothing new for perfect k. For imperfect k it is not evident that pseudo-reductive groups should admit a structure theory akin to that of reductive groups, especially in a form that is useful over arithmetically interesting fields. Over imperfect fields there are many non-reductive pseudo-reductive groups:

Example 1.1.3. The most basic example of a pseudo-reductive group over a field k is the Weil restriction $\mathrm{R}_{k'/k}(G')$ for a finite extension of fields k'/k and a connected reductive k'-group G' [CGP, Prop. 1.1.0]. If k'/k is not separable and $G' \neq 1$ then this k-group is not reductive (see

[CGP, Ex. 1.1.3, Ex. 1.6.1]). By [CGP, Prop. 1.2.3] solvable pseudo-reductive groups are always commutative (as in the connected reductive case), but they can fail to be tori and can even have nontrivial étale p-torsion in characteristic $p > 0$ [CGP, Ex. 1.6.3]. It seems very difficult to describe the structure of commutative pseudo-reductive groups.

The derived group of a pseudo-reductive k-group is pseudo-reductive (as is any smooth connected normal k-subgroup; check over k_s), so Example 1.1.1 shows that for imperfect k there are perfect pseudo-reductive k-groups that are *not* k-isomorphic to an isogenous quotient of $\mathrm{R}_{k'/k}(G')$ for any pair $(G', k'/k)$ as above.

Pseudo-reductivity may seem uninteresting because it is poorly behaved with respect to standard operations that preserve reductivity: inseparable extension of the ground field, quotients by central finite subgroup schemes of multiplicative type (e.g., $\mathrm{R}_{k'/k}(\mathrm{SL}_p)/\mu_p$ is not pseudo-reductive when k'/k is purely inseparable of degree $p = \mathrm{char}(k)$; see [CGP, Ex. 1.3.5]), and quotients by smooth connected normal k-subgroups N, even with $N = \mathscr{D}(N)$ [CGP, Ex. 1.4.9, Ex. 1.6.4]. Although it is not a robust concept, we will show that pseudo-reductivity is theoretically useful. For example, it is very effective in support of the above Principle.

The crux is that [CGP] provides a structure theory for pseudo-reductive k-groups "modulo the commutative case" (assuming $[k : k^2] \leqslant 2$ when $\mathrm{char}(k) = 2$). More precisely, there is a non-obvious procedure that constructs *all* pseudo-reductive k-groups from two ingredients: Weil restrictions $\mathrm{R}_{k'/k}(G')$ for connected semisimple G' over finite (possibly inseparable) extensions k'/k, and commutative pseudo-reductive k-groups. Such commutative groups turn out to be Cartan k-subgroups (i.e., centralizers of maximal k-tori).

1.2 Class numbers and Tate–Shafarevich sets

Now we turn to arithmetic topics. Let G be an affine group scheme of finite type over a global field k. Let S be a finite set of places of k containing the set S_∞ of archimedean places, and let $\mathbf{A}_k$ be the locally compact adele ring of k. For $k_S := \prod_{v \in S} k_v$, consider the double coset space

$$\Sigma_{G,S,K} := G(k)\backslash G(\mathbf{A}_k)/G(k_S)K = G(k)\backslash G(\mathbf{A}_k^S)/K \tag{1.2.1}$$

with $\mathbf{A}_k^S$ the factor ring of adeles (a_v) such that $a_v = 0$ for all $v \in S$ (so $\mathbf{A}_k = k_S \times \mathbf{A}_k^S$ as topological rings) and K a compact open subgroup of $G(\mathbf{A}_k^S)$. These double coset spaces arise in many contexts, such as labeling the connected components of Shimura varieties when k is a number field, classifying the dichotomy between global and everywhere-local conjugacy of rational points on k-schemes equipped with an action by an affine algebraic k-group, and studying the fibers of the localization map

$$\theta_{S,G'} : \mathrm{H}^1(k, G') \to \prod_{v \notin S} \mathrm{H}^1(k_v, G')$$

for affine algebraic k-groups G'. (This map also makes sense when the requirement $S \supseteq S_\infty$ is dropped.)

Remark 1.2.1. For any field k and k-group scheme G locally of finite type, the cohomology set $\mathrm{H}^1(k, G)$ is defined to be the pointed set of isomorphism classes of right G-torsors over k for the *fppf* topology. All such torsors in the *fppf* sheaf sense arise from schemes. (Proof: By [EGA, II, 6.6.5], translation arguments, and effective descent for quasi-projective schemes relative to finite extensions k'/k [SGA1, VIII, 7.7], it suffices to prove that G^0 is quasi-projective. The quasi-projectivity follows from [SGA3, $\mathrm{VI_A}$, 2.4.1] and [CGP, Prop. A.3.5].) We work with right

G-torsors rather than left G-torsors for consistency with the use of right actions in the definition of principal homogeneous spaces in [Se2, I, §5.2–§5.3]. It is equivalent to use torsors for the étale topology when G is smooth. Also, for smooth commutative G and any $m \geqslant 1$, the higher derived functors $\mathrm{H}^m_{\text{ét}}(k, G)$ and $\mathrm{H}^m_{\text{fppf}}(k, G)$ naturally coincide [BrIII, 11.7(1)]; this is useful when G is commutative and we wish to compute p-torsion in cohomology with $p = \mathrm{char}(k) > 0$.

Borel proved the finiteness of (1.2.1) when k is a number field [Bo1, Thm. 5.1]. His proof used archimedean places via the theory of Siegel domains developed earlier with Harish-Chandra. Another method due to Borel and G. Prasad works for all global fields when G is reductive (assuming $S \neq \emptyset$ in the function field case). But it is natural to consider *non-reductive* G. One reason is that if a connected semisimple k-group H acts on a k-scheme X then the study of local-to-global finiteness properties for the H-orbits on X leads to finiteness questions for double cosets as in (1.2.1) using the stabilizer group schemes $G = H_x$ at points $x \in X(k)$. Such stabilizers can be very bad even when smooth, as we saw in Example 1.1.1. Here is another kind of badness:

Example 1.2.2. For $H = \mathrm{R}_{k'/k}(\mathrm{SL}_N)$ acting on itself by conjugation and "generic" unipotent $x \in H(k) = \mathrm{SL}_N(k')$, $H_x = \mathrm{R}_{k'/k}(\mu_N \times U)$ for a k-split smooth connected unipotent U. If $N = p = \mathrm{char}(k)$ then H_x is not k-smooth; if also k'/k is purely inseparable of degree p then H_x is nonetheless *reduced* [CGP, Ex. A.8.3].

We conclude that it is reasonable to want (1.2.1) to be finite for any affine group scheme G of finite type over a global function field, using any finite $S \neq \emptyset$.

Some local-to-global orbit problems for actions by semisimple groups on schemes over a global field k reduce to the finiteness of Tate–Shafarevich sets $Ш^1_S(k, G') = \ker \theta_{S,G'}$ for *affine* algebraic k-groups G' that may not be reductive (or not smooth when $\mathrm{char}(k) > 0$). Finiteness of $Ш^1_S(k, G')$ was proved for any G' by Borel and Serre when $\mathrm{char}(k) = 0$ [BS, Thm. 7.1]. The case $\mathrm{char}(k) > 0$ was settled for reductive G' and solvable (smooth) G' by Borel–Prasad [BP, §4] and Oesterlé [Oes, IV, 2.6(a)] respectively; this is insufficient to easily deduce the general case (even for smooth G') since global function fields are imperfect.

1.3 Main results

Our first main result, upon which the others rest, is a generalization to nonzero characteristic of Borel's finiteness theorem for (1.2.1) over number fields. For $G = \mathrm{GL}_1$ over a number field and suitable K, the sets $\Sigma_{G,S_\infty,K}$ are the generalized ideal class groups of k. Thus, for any global field k we say G *has finite class numbers* if $\Sigma_{G,S,K}$ is finite for every non-empty finite S that contains S_∞ and every (equivalently, one) compact open subgroup $K \subseteq G(\mathbf{A}_k^S)$.

Theorem 1.3.1. (Finiteness of class numbers) *Let k be a global function field. Every affine k-group scheme G of finite type has finite class numbers.*

The absence of smoothness in Theorem 1.3.1 is easy to overcome with a trick (even though G_{red} may not be a k-subgroup of G, and when it is a k-subgroup it may not be smooth [CGP, Ex. A.8.3]), so the real work is in the smooth case. Likewise, it is elementary to reduce to the smooth connected case (see §3.2).

Example 1.3.2. Here is a proof that for global fields k, all smooth connected commutative affine k-groups G have finite class numbers. Let $T \subseteq G$ be the maximal k-split torus and $\overline{G} = G/T$. For any finite non-empty set S of places of k containing S_∞, the map $G(\mathbf{A}_k^S) \to \overline{G}(\mathbf{A}_k^S)$ is open since $G \twoheadrightarrow \overline{G}$ is smooth with connected kernel. Thus,

$$1 \to T(k)\backslash T(\mathbf{A}_k^S) \to G(k)\backslash G(\mathbf{A}_k^S) \to \overline{G}(k)\backslash \overline{G}(\mathbf{A}_k^S) \to 1$$

is exact (by Hilbert 90) with an open map on the right. The left term $T(k)\backslash T(\mathbf{A}_k^S)$ is compact since T is a k-split torus and GL_1 has finite class numbers.

It suffices to prove the finiteness of class numbers for $\overline{G}$, so we can assume that G does not contain GL_1 as a k-subgroup. Hence, G has no nontrivial k-rational characters $G \to \mathrm{GL}_1$ because such a character would map a maximal k-torus of G onto GL_1 [Bo2, 11.14] (forcing k-isotropicity, a contradiction). Since G is solvable and has no nontrivial k-rational characters, by a compactness result of Godement–Oesterlé [Oes, IV, 1.3] the coset space $G(k)\backslash G(\mathbf{A}_k)$ is compact and so G has finite class numbers.

As an application of Theorem 1.3.1 and the main results in [CGP], we establish the following analogue of a result of Borel and Serre [BS, Thm. 7.1, Cor. 7.12] over number fields:

Theorem 1.3.3. (Finiteness of Ш and local-to-global obstruction spaces) *Let k be a global function field and S a finite (possibly empty) set of places of k. Let G be an affine k-group scheme of finite type.*

(i) *The natural localization map $\theta_{S,G} : \mathrm{H}^1(k, G) \to \prod_{v \notin S} \mathrm{H}^1(k_v, G)$ has finite fibers. In particular, $Ш^1_S(k, G) := \ker \theta_{S,G}$ is finite.*

(ii) *Let X be a k-scheme equipped with a right action by G. For $x \in X(k)$, the set of points $x' \in X(k)$ in the same $G(k_v)$-orbit as x in $X(k_v)$ for all $v \notin S$ consists of finitely many $G(k)$-orbits.*

As with Theorem 1.3.1, the proof of Theorem 1.3.3 is easily reduced to the case of smooth G. The finiteness of $Ш^1_S(k, G)$ for smooth connected commutative affine k-groups G was proved by Oesterlé over all global fields by a uniform method [Oes, IV, 2.6(a)].

Remark 1.3.4. In Theorem 1.3.3 we cannot assume G is smooth in (i) because the proof of (ii) uses (i) for the scheme-theoretic stabilizer G_x at points $x \in X(k)$. By Examples 1.1.1 and 1.2.2, if $\mathrm{char}(k) > 0$ then G_x can be non-smooth even when G is semisimple or G_x is reduced, and even in cases with semisimple G and smooth G_x it can happen that the (unipotent) radical of $(G_x)_{\overline{k}}$ is not defined over k (inside of G_x).

The main arithmetic ingredient in the proof of Theorem 1.3.3 (in addition to Theorem 1.3.1) is Harder's vanishing theorem [Ha2, Satz A] for $\mathrm{H}^1(k, G)$ for any global function field k and any (connected and) simply connected semisimple k-group G. (This vanishing fails in general for number fields k with a real place.)

Remark 1.3.5. In the literature (e.g., [Mi2, I], [Ma, §16]), the notations $Ш^1_S$ and $Ш_S$ are used for other definitions resting on Galois cohomology or flat cohomology over the S-integers. For abelian varieties and their Néron models these definitions are related to $Ш^1_S$ as in Theorem 1.3.3(i), but we do not use them.

Finally, we turn to the topic of volumes. In [Oes, I, 4.7], the *Tamagawa measure* μ_G on $G(\mathbf{A}_k)$ is defined for any smooth affine group G over a global field k. Letting $\|\cdot\|_k : \mathbf{A}_k^\times \to \mathbf{R}_{>0}^\times$ be the idelic norm, define $G(\mathbf{A}_k)^1$ to be the closed subgroup of points $g \in G(\mathbf{A}_k)$ such that $\|\chi(g)\|_k = 1$ for all k-rational characters χ of G (so $G(k) \subseteq G(\mathbf{A}_k)^1$, and $G(\mathbf{A}_k)^1 = G(\mathbf{A}_k)$ if G has no nontrivial k-rational characters). This is a unimodular group [Oes, I, 5.8]. Now assume G is connected. The Tamagawa measure is used in [Oes, I, 5.9] to define a canonical measure μ_G^1 on $G(\mathbf{A}_k)^1$, so by unimodularity there is an induced measure on the quotient space $G(k)\backslash G(\mathbf{A}_k)^1$ (or equivalently, on the quotient space $G(\mathbf{A}_k)^1/G(k)$). The volume τ_G of this quotient space is the *Tamagawa number* for G; it is not evident from the definition if this is finite.

The finiteness of τ_G for general (smooth connected affine) G was proved over number fields by Borel; it was proved over function fields in the reductive case by Harder and in the solvable case by Oesterlé (see [Oes, I, 5.12] for references). The results of Harder and Oesterlé are insufficient to easily deduce the finiteness of τ_G in all cases over global function fields (e.g., (1.1.1) is generally missing).

For smooth connected affine groups over global fields, Oesterlé [Oes, II, III] worked out the behavior of Tamagawa numbers with respect to short exact sequences and Weil restriction through finite (possibly inseparable) extension fields, including the behavior of finiteness properties for Tamagawa numbers relative to these situations. His formulas for the behavior under short exact sequences [Oes, III, 5.2, 5.3] were conditional on the finiteness of certain auxiliary Tate–Shafarevich sets and analogues of class numbers (which he did not know to always be finite). Our results (Theorem 1.3.3(i) and a variant on Theorem 1.3.1 with $S = \emptyset$ given in Corollary 7.3.5) establish these finiteness hypotheses in general, so by combining Oesterlé's work with the structure theory of pseudo-reductive groups from [CGP] we can prove the function field version of Borel's general finiteness theorem for τ_G:

Theorem 1.3.6. (Finiteness of Tamagawa numbers) *For any smooth connected affine group G over a global function field, the Tamagawa number τ_G is finite.*

Remark 1.3.7. Let $1 \to G' \to G \to G'' \to 1$ be an exact sequence of smooth connected affine groups over a global field k, and assume $G(\mathbf{A}_k) \to G''(\mathbf{A}_k)$ has normal image (e.g., G' central in G, or $\mathrm{char}(k) = 0$ [Oes, III, 2.4]). Oesterlé's formula for $\tau_G/(\tau_{G'}\tau_{G''})$ over number fields in [Oes, III, 5.3] is valid unconditionally when $\mathrm{char}(k) > 0$, by Theorem 1.3.3(i) and Corollary 7.3.5.

Going beyond the affine case, it was conditionally proved by Mazur [Ma, §15–§17] over number fields k (assuming the finiteness of Tate–Shafarevich groups $\text{Ш}^1_\emptyset(k, A)$ for abelian varieties A over k) that Theorem 1.3.3 holds for $S = \emptyset$ with any k-group scheme G locally of finite type for which the geometric component group $(G/G^0)(k_s) = G(k_s)/G^0(k_s)$ satisfies certain group-theoretic finiteness properties. In §7.5 we use Theorem 1.3.3 to prove an analogous result over global function fields k. Mazur's proof does not work in nonzero characteristic (for reasons we explain after Example 7.5.1), so we use another argument that also works over number fields and relies on additional applications of [CGP] over global function fields.

1.4 Strategy of proof of Theorem 1.3.1

If $1 \to G' \to G \to G'' \to 1$ is an exact sequence of smooth connected affine groups over a global field k, then the open image of $G(\mathbf{A}_k) \to G''(\mathbf{A}_k)$ can fail to have finite index, even if G' is a torus (e.g., take $G \to G''$ to be the norm $\mathrm{R}_{k'/k}(\mathrm{GL}_1) \to \mathrm{GL}_1$ for a quadratic Galois extension k'/k). The same problem can occur for $G(k_S) \to G''(k_S)$ when $\mathrm{char}(k) > 0$ if G' is unipotent but not k-split [CGP, Ex. 11.3.3]. Over global function fields, it is a serious problem to overcome such difficulties.

A well-known strategy to bypass some of these problems is to find a presentation of G that allows us to exploit the cohomological and arithmetic properties of simply connected semisimple groups. Let us recall how this goes in the familiar case of a connected reductive group G over a global field k. The so-called z-construction (reviewed in §5.1) produces a diagram of short exact sequences

(1.4.1)

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathscr{D}(E) & & & & \\ & & & & \downarrow & & & & \\ 1 & \longrightarrow & T' & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & & & \downarrow & & & & \\ & & & & T'' & & & & \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \end{array}$$

in which T' and T'' are tori and E is a connected reductive k-group such that the semisimple derived group $\mathscr{D}(E)$ is simply connected and T' has trivial degree-1 Galois cohomology over k and its completions. By strong approximation for (connected and) simply connected semisimple groups (and a compactness argument in the k-anisotropic case), $\mathscr{D}(E)$ has finite class numbers. By theorems of Kneser-Bruhat-Tits [BTIII, Thm. 4.7(ii)] and Harder [Ha2, Satz A], the degree-1 Galois cohomology of (connected and) simply connected semisimple groups over non-archimedean local fields and global function fields vanishes. Thus, finiteness of class numbers for E can be deduced from the cases of $\mathscr{D}(E)$ and the commutative T'' when $\mathrm{char}(k) > 0$. Finiteness for G follows from that of E via (1.4.1) due to the vanishing of degree-1 Galois cohomology for T'.

Adapting the z-construction beyond the reductive case is non-trivial when $\mathrm{char}(k) > 0$; this is done by using the structure theory from [CGP] for pseudo-reductive groups. There are several ways to carry it out, depending on the circumstances, and in the role of T' we sometimes use a solvable smooth connected affine k-group whose local Galois cohomology in degree 1 is infinite. To overcome such infinitude problems we use a toric criterion for an open subgroup of $G(L)$ to have finite index when L is a non-archimedean local field and G is a smooth connected affine L-group that is "quasi-reductive" in the sense of Bruhat and Tits [BTII, 1.1.12] (i.e., G has no nontrivial smooth connected unipotent normal L-subgroup that is L-split). The proof of this criterion (Proposition 4.1.9) also rests on the structure theory from [CGP].

1.5 Overview

Let us now give an overview of the paper. The general structure theorems from [CGP] are recorded in §2 in a form sufficient for our needs. In §3, which involves no novelty, we adapt arguments of Borel over number fields from [Bo1, §1] to show that a smooth affine group over a global field has finite class numbers if its identity component does. In §4 we recall (for ease of later reference) some well-known finiteness properties of tori over local fields and of adelic coset spaces, and record some generalizations.

In §5 we use the structure theory for pseudo-reductive groups to prove Theorem 1.3.1 for pseudo-reductive groups over global function fields via reduction to the known case of (connected and) simply connected semisimple groups. We prove the smooth case of Theorem 1.3.1 by reduction to the pseudo-reductive case. Although the underlying reduced scheme of an affine finite type k-group is generally not k-smooth (nor even a k-subgroup) when k is a global function field, there is a trick that enables us to reduce the proof of Theorem 1.3.1 to the case when G is smooth. This trick is also useful in the proof of Theorem 1.3.3 because (as we noted in Remark 1.3.4) the proof of part (ii) of this theorem requires part (i) for the isotropy group scheme $G_x \subseteq G$ that may be non-smooth even if G is smooth.

In §6 we prove Theorem 1.3.3 as an application of Theorem 1.3.1 and the structure of pseudo-reductive groups. In §7 we give applications of Theorem 1.3.3, including Theorem 1.3.6 and an extension of Theorem 1.3.3(i) to non-affine k-groups conditional on the Tate–Shafarevich conjecture for abelian varieties. A difficulty encountered here is that Chevalley's well-known theorem expressing a smooth connected group over a perfect field as an extension of an abelian variety by a smooth connected affine group is *false* over every imperfect field.

In Appendix A we prove a technical result on properness of a certain map between adelic coset spaces. This is used in §5, and in §A.5 we combine it with results from [CGP] to give the first general proof of the sufficiency of the function field analogue of a compactness criterion of Godement for certain adelic coset spaces over number fields; see Theorem A.5.5(i). (The necessity of Godement's criterion is proved in [Oes, IV, 1.4], and sufficiency was previously known in the semisimple and solvable cases.) We also prove a local analogue of Godement's criterion (Proposition A.5.7). In Appendix B we review (as a convenient reference) how to generalize the low-degree cohomology of smooth algebraic groups [Se2, I, §5] to the case of general group schemes of finite type over a field, especially the twisting method and the necessity of computing degree-2 commutative cohomology in terms of gerbes rather than via Čech theory in the non-smooth case.

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1.7 Notation and Terminology

We make no connectivity assumptions on group schemes. If G is an affine group scheme of finite type over a field k then $\mathrm{X}_k(G)$ denotes the character group $\mathrm{Hom}_k(G, \mathrm{GL}_1)$ over k; this is a finitely generated $\mathbf{Z}$-module (and torsion-free when G is smooth and connected).

The theory of forms of smooth connected unipotent groups over imperfect fields is very subtle (even for k-forms of $\mathbf{G}_{\mathrm{a}}$; see [Ru]). We require facts from that theory that are not widely known, and refer to [CGP, App. B] for an account of Tits' important work on this topic (including what is required in [Oes], whose results we use extensively).

A smooth connected unipotent group U over a field k is *k-split* if it admits a composition series by smooth connected k-subgroups with successive quotients k-isomorphic to $\mathbf{G}_{\mathrm{a}}$. The k-split property is inherited by arbitrary quotients [Bo2, 15.4(i)], and every smooth connected unipotent k-group is k-split when k is perfect [Bo2, 15.5(ii)]. Beware that (in contrast with tori) the k-split property in the unipotent case is not inherited by smooth connected normal k-subgroups when k is not perfect. For example, if $\mathrm{char}(k) = p > 0$ and $a \in k$ is not in k^p then $y^p = x - ax^p$ is a k-subgroup of the k-split $\mathbf{G}_{\mathrm{a}} \times \mathbf{G}_{\mathrm{a}}$ and it is a k-form of $\mathbf{G}_{\mathrm{a}}$ that is not k-split. (Its regular compactification $y^p = xz^{p-1} - az^p$ has no k-rational point at infinity.)

If $A \to A'$ is a map of rings and Z is a scheme over A then $Z_{A'}$ denotes the base change of Z to an A'-scheme. If Y is a scheme, then Y_{red} denotes the underlying reduced scheme.

We use scheme-theoretic Weil restriction of scalars (in the quasi-projective case) with respect to possibly inseparable finite extensions of the base field (as well as a variant for base rings). For a development of Weil restriction in the context of schemes we refer the reader to [Oes, App. 2, 3], [BLR, §7.6], and [CGP, §A.5, §A.7]. If k is a field and k' is a nonzero finite reduced k-algebra (i.e., a product of finitely many finite extension fields of k) then $\mathrm{R}_{k'/k}$ denotes the Weil restriction functor from quasi-projective k'-schemes to (quasi-projective) k-schemes. If k'/k is a finite separable field extension then this functor coincides with the Galois descent construction as used in [We] and many other works on algebraic groups.

We shall need to use the equivalent but different approaches of Weil and of Grothendieck for adelizing separated schemes of finite type over global fields, and we use without comment the functorial properties of these constructions (e.g., good behavior with respect to Weil restriction of scalars and smooth surjective maps with *geometrically connected* fibers). This material is "well-known" (cf. [CS, p. 87]), and we refer to [Oes, I, 3.1] and [C2] for a detailed discussion.

A diagram $1 \to G' \to G \to G'' \to 1$ of group schemes of finite type over a noetherian scheme is a *short exact sequence* if $G \to G''$ is faithfully flat with scheme-theoretic kernel G'; e.g., we use this over rings of S-integers of global fields. Non-smooth group schemes naturally arise in our arguments, even in the study of smooth groups (e.g., kernels may not be smooth), so we will need to form quotients modulo non-smooth normal subgroups.

For any finite type group scheme G and normal closed subgroup scheme N over a field F, the F-group G/N is taken in the sense of Grothendieck; see [SGA3, VI, 3.2(iv), 5.2]. We now make some comments on the quotient process over F, for the benefit of readers who are more comfortable with smooth groups. In general the quotient map $G \to G/N$ is faithfully flat with the expected universal property for N-invariant maps from G to arbitrary F-schemes, and its formation commutes with any extension on F. If G is F-smooth then G/N is F-smooth even if N is not (since we can assume F is algebraically closed, and regularity descends through faithfully flat extensions of noetherian rings). By [SGA3, $\mathrm{VI_B}$, 11.17], G/N is affine when G is affine. If G is smooth and affine and N is smooth then G/N coincides with the concept of quotient used in textbooks on linear algebraic groups, as both notions of quotient satisfy the same universal property.

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2. Pseudo-reductive groups

Recall from §1.1 that a *pseudo-reductive* group G over a field k is a smooth connected affine k-group whose only smooth connected unipotent normal k-subgroup is $\{1\}$. A smooth connected affine k-group G is *pseudo-simple* (over k) if G is non-commutative and has no nontrivial smooth connected normal proper k-subgroup. Finally, G is *absolutely pseudo-simple* over k if G_{k_s} is pseudo-simple over k_s. By [CGP, Lemma 3.1.2], G is absolutely pseudo-simple over k if and only if the following three conditions hold: (i) G is pseudo-reductive over k, (ii) $G = \mathscr{D}(G)$, and (iii) $G^{\mathrm{ss}}_{\overline{k}}$ is simple.

Below we discuss a general structure theorem for pseudo-reductive groups over an arbitrary (especially imperfect) field k, assuming $[k : k^2] \leqslant 2$ when $\mathrm{char}(k) = 2$. The case of most interest to us will be when k is a local or global function field (so $[k : k^2] = 2$ when $\mathrm{char}(k) = 2$), but the results that we are about to describe are no easier to prove in these cases than in general.

2.1 Standard pseudo-reductive groups

The following pushout construction provides a large class of pseudo-reductive groups.

Example 2.1.1. Let k' be a nonzero finite reduced k-algebra and let G' be a k'-group whose fiber over each factor field of k' is connected and reductive. Let $T' \subseteq G'$ be a maximal k'-torus, $Z_{G'}$ the (scheme-theoretic) center of G', and $\overline{T}' = T'/Z_{G'}$. The left action of T' on G' via conjugation factors through a left action of $\overline{T}'$ on G', so $\mathrm{R}_{k'/k}(\overline{T}')$ acts on $\mathrm{R}_{k'/k}(G')$ on the left via functoriality. It can happen (e.g., if k' is a nontrivial purely inseparable extension field of k and $Z_{G'}$ is not k'-étale) that $\mathrm{R}_{k'/k}(T') \to \mathrm{R}_{k'/k}(\overline{T}')$ is *not* surjective.

By [CGP, Prop. A.5.15], the k-group $\mathrm{R}_{k'/k}(T')$ is a Cartan k-subgroup of the pseudo-reductive k-group $\mathrm{R}_{k'/k}(G')$ (i.e., it is the centralizer of a maximal k-torus). Its conjugation action on $\mathrm{R}_{k'/k}(G')$ factors as the composition of the natural homomorphism $\mathrm{R}_{k'/k}(T') \to \mathrm{R}_{k'/k}(\overline{T}')$ and the natural left action of $\mathrm{R}_{k'/k}(\overline{T}')$. Now the basic idea is to try to "replace" the Cartan k-subgroup $\mathrm{R}_{k'/k}(T')$ with another commutative pseudo-reductive k-group C that acts on $\mathrm{R}_{k'/k}(G')$ through a k-homomorphism to $\mathrm{R}_{k'/k}(\overline{T}')$.

To make the idea precise, suppose that there is given a factorization

$$\mathrm{R}_{k'/k}(T') \xrightarrow{\phi} C \to \mathrm{R}_{k'/k}(\overline{T}') \tag{2.1.1}$$

of the Weil restriction to k of the canonical projection $T' \to \overline{T}'$ over k', with C a commutative pseudo-reductive k-group; it is *not* assumed that ϕ is surjective. We let C act on $\mathrm{R}_{k'/k}(G')$ on the left through its map to $\mathrm{R}_{k'/k}(\overline{T}')$ in (2.1.1), so there arises a semidirect product group $\mathrm{R}_{k'/k}(G') \rtimes C$. Using the pair of homomorphisms

$$j : \mathrm{R}_{k'/k}(T') \hookrightarrow \mathrm{R}_{k'/k}(G'), \ \ \phi : \mathrm{R}_{k'/k}(T') \to C,$$

consider the twisted diagonal map

$$\alpha : \mathrm{R}_{k'/k}(T') \to \mathrm{R}_{k'/k}(G') \rtimes C \tag{2.1.2}$$

defined by $t' \mapsto (j(t')^{-1}, \phi(t'))$. This is easily seen to be an isomorphism onto a central subgroup. The resulting quotient $G = \mathrm{coker}(\alpha)$ is a kind of non-commutative pushout that replaces $\mathrm{R}_{k'/k}(T')$ with C. By [CGP, Prop. 1.4.3], it is pseudo-reductive over k (since C is pseudo-reductive).

Definition 2.1.2. A *standard* pseudo-reductive k-group is a k-group scheme G isomorphic to a k-group $\mathrm{coker}(\alpha)$ arising from the pushout construction in Example 2.1.1.

If the map ϕ in (2.1.1) is surjective then the k-group $G = \mathrm{coker}(\alpha)$ is the quotient of $\mathrm{R}_{k'/k}(G')$ modulo a k-subgroup scheme $Z := \ker\phi \subseteq \mathrm{R}_{k'/k}(Z_{G'})$. Beware that in general not every quotient of $\mathrm{R}_{k'/k}(G')$ modulo a k-subgroup scheme Z of $\mathrm{R}_{k'/k}(Z_{G'})$ is pseudo-reductive over k. (By [CGP, Rem. 1.4.6], $\mathrm{R}_{k'/k}(G')/Z$ is pseudo-reductive over k if and only if the commutative $C := \mathrm{R}_{k'/k}(T')/Z$ is pseudo-reductive.) At the other extreme, if G' is trivial then $G = C$ is an arbitrary commutative pseudo-reductive k-group.

By [CGP, Rem. 1.4.2], if G is a standard pseudo-reductive k-group constructed from data $(G', k'/k, T', C)$ as in Example 2.1.1 then C is a Cartan k-subgroup of G. This Cartan k-subgroup is generally not a k-torus, in contrast with the case of connected reductive groups. In fact, by [CGP, Thm. 11.1.1], if $\mathrm{char}(k) \neq 2$ then a pseudo-reductive k-group is reductive if and only if its Cartan k-subgroups are tori; this equivalence lies quite deep (e.g., its proof rests on nearly everything in [CGP]), and it is false over every imperfect field of characteristic 2 (even in the standard case; see [CGP, Ex. 11.1.2]).

2.2 Standard presentations

There is a lot of flexibility in the choice of $(G', k'/k, T', C)$ and the diagram (2.1.1) giving rise to a fixed standard pseudo-reductive k-group G. In [CGP, Thm. 4.1.1] it is shown that if G is a non-commutative standard pseudo-reductive k-group then it arises via the construction in Example 2.1.1 using a 4-tuple $(G', k'/k, T', C)$ for which the fibers of G' over the factor fields of k' are semisimple, *absolutely simple*, and *simply connected*. Under these properties, the map $j : \mathrm{R}_{k'/k}(G') \to G$ with central kernel $\ker \phi$ has image $\mathscr{D}(G)$ due to the simply connected condition on G' [CGP, Cor. A.7.11], and the triple $(G', k'/k, j)$ is uniquely determined by G up to unique k-isomorphism [CGP, Prop. 4.2.4, Prop. 5.1.7(1)].

By [CGP, Prop. 4.1.4], the triple $(G', k'/k, j)$ corresponding to such (non-commutative) G satisfies the following properties. There is a natural bijection between the set of maximal k-tori $T \subset G$ and the set of maximal k'-tori $T' \subset G'$, for each such matching pair (T, T') there is a diagram (2.1.1) that (together with $(G', k'/k)$) gives rise to G via the pushout construction in Example 2.1.1, and the commutative pseudo-reductive k-group C in the associated diagram (2.1.1) is identified with the Cartan k-subgroup $Z_G(T)$ in G.

For a non-commutative standard pseudo-reductive k-group G, there is a uniqueness property for the diagram (2.1.1) in terms of the above canonically associated $(G', k'/k, j)$ and the choice of T. This is stated precisely in [CGP, Prop. 4.1.4(3)], and here we record an important consequence from [CGP, Prop. 5.2.2]: the 4-tuple $(G', k'/k, T', C)$ is (uniquely) *functorial* with respect to k-isomorphisms in the pair (G, T). This 4-tuple is called the *standard presentation* of G adapted to the choice of T, suppressing the mention of the factorization diagram (2.1.1) that is an essential ingredient in the usefulness of this concept.

2.3 Structure theorems for pseudo-reductive groups

Any connected reductive k-group G is standard (use $k' = k$, $G' = G$, and $C = T'$), as is any commutative pseudo-reductive k-group (use $k' = k$, $G' = 1$, and $C = G$). It is difficult to say much about the general structure of commutative pseudo-reductive groups, but the commutative case is essentially the only mystery. This follows from the ubiquity of the pseudo-reductive k-groups arising via Example 2.1.1, modulo some complications when $\mathrm{char}(k) \in \{2, 3\}$, as we now explain.

Let G be a pseudo-reductive group, and T a maximal k-torus in G. The set of weights for T_{k_s} acting on $\mathrm{Lie}(G_{k_s})$ naturally forms a root system [CGP, §3.2], but this may be non-reduced. (If G is a standard pseudo-reductive group then this root system is always reduced [CGP, Ex. 2.3.2, Prop. 2.3.15].) The cases with a non-reduced root system can only exist when k is imperfect and $\mathrm{char}(k) = 2$ [CGP, Thm. 2.3.10], and conversely for any imperfect k with $\mathrm{char}(k) = 2$ and any integer $n \geqslant 1$ there exists (G, T) over k such that the associated root system is non-reduced and $\dim T = n$ [CGP, Thm. 9.3.10].

Before we can state the general classification theorems for pseudo-reductive groups (in all characteristics), we need to go beyond the standard case by introducing Tits' constructions of additional absolutely pseudo-simple groups $\mathscr{G}$ over imperfect fields k of characteristic 2 or 3. There are two classes of such constructions, depending on whether or not the root system associated to $\mathscr{G}_{k_s}$ is reduced or non-reduced. First we take up the cases with a reduced root system.

Let k be an arbitrary field of characteristic $p \in \{2,3\}$, and let G be a connected semisimple k-group that is absolutely simple and simply connected with Dynkin diagram having an edge with multplicity p (i.e., type G_2 when $p=3$, and type B_n, C_n ($n \geqslant 2$), or F_4 when $p=2$). By [CGP, Lemma 7.1.2], the relative Frobenius isogeny $G \to G^{(p)}$ admits a unique nontrivial factorization in k-isogenies

$$G \xrightarrow{\pi} \overline{G} \to G^{(p)} \tag{2.3.1}$$

such that π is non-central and has no nontrivial factorization; π is the *very special k-isogeny* for G, and $\overline{G}$ is the *very special quotient* of G. The p-Lie algebra of the height-1 normal k-subgroup scheme $\ker \pi$ in G is the unique non-central G-stable Lie subalgebra of $\mathrm{Lie}(G)$ that is irreducible under the adjoint action of G [CGP, Lemma 7.1.2]. The connected semisimple k-group $\overline{G}$ is also simply connected, with type dual to that of G [CGP, Prop. 7.1.5].

Now also assume k is imperfect and let k'/k be a nontrivial finite extension such that $k'^p \subseteq k$. Let G' be a connected semisimple k'-group that is absolutely simple and simply connected with Dynkin diagram having an edge with multiplicity p. Let $\pi' : G' \to \overline{G}'$ be the very special k'-isogeny. The Weil restriction $f := \mathrm{R}_{k'/k}(\pi')$ of π' is not an isogeny since $k' \neq k$. (Its kernel is non-smooth with dimension > 0.)

Definition 2.3.1. Let k be an imperfect field of characteristic $p \in \{2,3\}$. A k-group scheme $\mathscr{G}$ is called a *basic exotic pseudo-reductive* k-group if there exists a pair $(G', k'/k)$ as above and a Levi k-subgroup $\overline{G} \subseteq \mathrm{R}_{k'/k}(\overline{G}')$ such that $\mathscr{G}$ is k-isomorphic to the scheme-theoretic preimage $f^{-1}(\overline{G}) \subseteq \mathrm{R}_{k'/k}(G')$ as a k-group and $f^{-1}(\overline{G})_{k_s}$ contains a Levi k_s-subgroup of $\mathrm{R}_{k'/k}(G')_{k_s}$.

Applying [CGP, Lemma 7.2.1, Thm. 7.2.3] over k_s, any k-group $\mathscr{G}$ as in Definition 2.3.1 is pseudo-reductive (hence connected and k-smooth). Moreover, by [CGP, Prop. 7.2.7(1),(2)] the k-group $\mathscr{G}$ satisfies the following properties: it is not reductive, $\mathscr{G}_{k_s}$ has a reduced root system, the triple $(G', k'/k, \overline{G})$ is uniquely determined by $\mathscr{G}$ up to a unique k-isomorphism, and the induced map $f : \mathscr{G} \to \overline{G}$ is surjective. By [CGP, Prop. 8.1.1, Cor. 8.1.3], such $\mathscr{G}$ are absolutely pseudo-simple and are never standard pseudo-reductive groups.

Examples exist in abundance: by [CGP, Thm. 7.2.3] any pair $(G', k'/k)$ as above with k'-split G' arises from some such $\mathscr{G}$. The odd-looking Levi subgroup condition over k_s at the end of Definition 2.3.1 cannot be dropped; see [CGP, Ex. 7.2.2, Prop. 7.3.1, Prop. 7.3.6] for the significance of this condition, as well as more natural-looking formulations of it. Basic exotic pseudo-reductive groups are used in the following generalization of the "standard construction" from Example 2.1.1.

Example 2.3.2. Let k be a field, k' a nonzero finite reduced k-algebra, and G' a k'-group with absolutely pseudo-simple fibers. For each factor field k'_i of k', assume that the k'_i-fiber G'_i of G' is either semisimple and simply connected or (if k is imperfect with $\mathrm{char}(k) \in \{2,3\}$) basic exotic in the sense of Definition 2.3.1. Let T' be a maximal k'-torus in G', and C' the associated Cartan k'-subgroup $Z_{G'}(T')$. By [CGP, Prop. A.5.15(3)] it follows that $\mathrm{R}_{k'/k}(C')$ is a Cartan k-subgroup of $\mathrm{R}_{k'/k}(G')$.

Consider a k-homomorphism $\phi : \mathrm{R}_{k'/k}(C') \to C$ to a commutative pseudo-reductive k-group C, and a left action of C on $\mathrm{R}_{k'/k}(G')$ whose composition with ϕ is the standard action and whose effect on the k-subgroup $\mathrm{R}_{k'/k}(C') \subset \mathrm{R}_{k'/k}(G')$ is trivial. We then obtain a semi-direct product $\mathrm{R}_{k'/k}(G') \rtimes C$ and (as in (2.1.2) in the standard construction) the anti-diagonal embedding

$$\mathrm{R}_{k'/k}(C') \to \mathrm{R}_{k'/k}(G') \rtimes C$$

is a central k-subgroup. Thus, it makes sense to form the quotient

$$G := (\mathrm{R}_{k'/k}(G') \rtimes C)/\,\mathrm{R}_{k'/k}(C').$$

The k-group G is pseudo-reductive [CGP, Prop. 1.4.3], and $\mathscr{D}(G)$ is the image of $\mathrm{R}_{k'/k}(G')$ [CGP, Cor. A.7.11, Prop. 8.1.2].

By [CGP, Prop. 10.2.2(1)], there is a unique maximal k-torus T in G that contains the image of the maximal k-torus of $\mathrm{R}_{k'/k}(C')$ under the composite map $\mathrm{R}_{k'/k}(C') \to \mathrm{R}_{k'/k}(G') \to G$, and $C = Z_G(T)$. In particular, C is a Cartan k-subgroup of G. Moreover, (G_{k_s}, T_{k_s}) has a reduced root system for the same reasons as in the standard case [CGP, Rem. 2.3.9].

Definition 2.3.3. A pseudo-reductive group G over a field k is *generalized standard* if it is commutative or isomorphic to the construction in Example 2.3.2 arising from some 4-tuple $(G', k'/k, T', C)$ as considered there. For non-commutative G this 4-tuple is called a *generalized standard presentation* of G adapted to the unique maximal k-torus T in the Cartan k-subgroup $C \subset G$. (By [CGP, Thm. 1.3.9], this recovers Definition 2.1.2 and §2.2 unless k is imperfect with $\mathrm{char}(k) \in \{2, 3\}$ and $G' \to \mathrm{Spec}\, k'$ has a basic exotic fiber.)

Remark 2.3.4. By [CGP, Prop. 10.2.4], the generalized standard presentation is (uniquely) functorial with respect to isomorphisms in (G, T). In this sense, the generalized standard presentation of G is uniquely determined by T. Moreover, by [CGP, Prop. 10.2.2(3)], if a non-commutative G admits a generalized standard presentation adapted to one choice of T then the same holds for any choice, so the "generalized standard" property is independent of T. Finally, in the non-commutative case, the triple $(G', k'/k, j)$ encoding the map $j : \mathrm{R}_{k'/k}(G') \to G$ is uniquely functorial with respect to isomorphisms in the k-group G [CGP, Rem. 10.1.11, Prop. 10.1.12(1)], so $(G', k'/k, j)$ is independent of the choice of generalized standard presentation of G.

Next we turn to the case of absolutely pseudo-simple G for which G_{k_s} has a non-reduced root system.

Definition 2.3.5. Assume k is imperfect with $\mathrm{char}(k) = 2$. A *basic non-reduced* pseudo-simple k-group is an absolutely pseudo-simple k-group G such that G_{k_s} has a non-reduced root system and the field of definition k'/k for $\mathscr{R}(G_{\overline{k}}) \subset G_{\overline{k}}$ is quadratic over k; we write $(G_{k'})^{\mathrm{ss}}$ to denote the k'-descent of $G_{\overline{k}}/\mathscr{R}(G_{\overline{k}})$ as a quotient of $G_{k'}$.

Theorem 2.3.6. *Let k be a field of characteristic 2 such that $[k : k^2] = 2$.*

(i) *For each $n \geqslant 1$, up to k-isomorphism there exists exactly one basic non-reduced pseudo-simple k-group for which the maximal k-tori have dimension n.*

(ii) *For a pseudo-reductive k-group G such that G_{k_s} has a non-reduced root system, there is a unique decomposition*

$$G = G_1 \times G_2 \tag{2.3.2}$$

such that $(G_2)_{k_s}$ has a reduced root system and $G_1 \simeq \mathrm{R}_{K/k}(\mathscr{G})$ for a pair $(\mathscr{G}, K/k)$ consisting of a nonzero finite reduced k-algebra K and a K-group $\mathscr{G}$ whose fibers are basic non-reduced pseudo-reductive groups over the factor fields of K. (The k-group G_2 may be trivial.)
Moreover, $(\mathscr{G}, K/k)$ is uniquely functorial with respect to k-isomorphisms in G_1, and if $\{K_i\}$ is the set of factor fields of K and $\mathscr{G}_i$ is the K_i-fiber of $\mathscr{G}$ then the smooth connected normal k-subgroups of G_1 are precisely the products among the k-subgroups $\mathrm{R}_{K_i/k}(\mathscr{G}_i)$. In particular, G_1 is perfect.

Proof. Part (i) is [CGP, Thm. 9.4.3(1)], and (2.3.2) is [CGP, Thm 5.1.1(3), Prop. 10.1.4(1)]. The uniqueness and properties of $(\mathscr{G}, K/k)$ are [CGP, Prop 10.1.4(2),(3)]. ∎

Remark 2.3.7. The uniqueness in Theorem 2.3.6(i) fails whenever $[k : k^2] > 2$. The construction of basic non-reduced pseudo-simple k-groups is very indirect, resting on the theory of birational group laws. There is an explicit description of the birational group law on an "open Bruhat cell" when $[k : k^2] = 2$ (see [CGP, Thm. 9.3.10(2)]). For our purposes this can be suppressed, due to Theorem 2.3.8(ii) below.

The decomposition in (2.3.2) shows that the general classification of pseudo-reductive k-groups, assuming $[k : k^2] \leqslant 2$ when $\mathrm{char}(k) = 2$, breaks into two cases: the case when G_{k_s} has a reduced root system, and the case when G is a basic non-reduced pseudo-simple k-group. The main classification theorem from [CGP] is:

Theorem 2.3.8. *Let G be a pseudo-reductive group over a field k, with $p := \mathrm{char}(k)$. If $p = 2$ then assume $[k : k^2] \leqslant 2$.*

(i) *If G_{k_s} has a reduced root system then the k-group G is generalized standard* (*so it is standard except possibly if k is imperfect with $p \in \{2, 3\}$*).

(ii) *Assume $p \in \{2, 3\}$, $[k : k^p] = p$, and either that G is a basic exotic pseudo-reductive k-group or $p = 2$ and G is a basic non-reduced pseudo-simple k-group.*
In the basic exotic case, there is a surjective k-homomorphism $f : G \twoheadrightarrow \overline{G}$ onto a connected semisimple k-group $\overline{G}$ that is absolutely simple and simply connected such that: (a) *the induced maps $G(k) \to \overline{G}(k)$ and $\mathrm{H}^1(k, G) \to \mathrm{H}^1(k, \overline{G})$ are bijective,* (b) *if T is a maximal k-torus* (*resp. maximal k-split k-torus*) *in G then the same holds for $\overline{T} := f(T)$ in $\overline{G}$ and $T \to \overline{T}$ is an isogeny,* (c) *the formation of f is functorial with respect to k-isomorphisms in G and commutes with separable extension on k,* (d) *if k is equipped with an absolute value* (*resp. is a global function field*) *then $G(k) \to \overline{G}(k)$* (*resp. $G(\mathbf{A}_k) \to \overline{G}(\mathbf{A}_k)$*) *is a homeomorphism.*
In the basic non-reduced pseudo-simple case the same holds using $\overline{G} = \mathrm{R}_{k^{1/2}/k}(\overline{G}')$ for a $k^{1/2}$-group $\overline{G}'$ that is functorial with respect to k-isomorphisms in G and is $k^{1/2}$-isomorphic to Sp_{2n}, where n is the dimension of maximal tori of G.

By [CGP, Thm. C.2.3], the maximal k-split k-tori in any smooth connected affine group H over a field k are $H(k)$-conjugate.

Proof. The assertion in (i) is [CGP, Thm 10.2.1(2), Prop. 10.2.4]. For (ii), we first dispose of the case when $p = 2$ and G is a basic non-reduced pseudo-simple k-group. Let $k' = k^{1/2}$ and $G' = (G_{k'})^{\mathrm{ss}} = (G_{k'})^{\mathrm{red}}$, and define $\xi_G : G \to \mathrm{R}_{k'/k}(G')$ to be the natural k-map (so $\ker \xi_G$ is a unipotent group scheme). By [CGP, Thm. 9.4.3(1)] we have $G' \simeq \mathrm{Sp}_{2n}$ as k'-groups for some $n \geqslant 1$, and by [CGP, Prop. 9.4.12(1)] the natural map $G(k) \to G'(k')$ is bijective and $\mathrm{H}^1(k, G) = 1$. Moreover, if k is topologized by an absolute value (resp. is a global function field) then $G(k) \to G'(k')$ is a homeomorphism (resp. $G(\mathbf{A}_k) \to G'(\mathbf{A}_{k'})$ is a homeomorphism) due to [CGP, Prop. 9.4.12(2),(3)]. Thus, if we take $\overline{G} = \mathrm{R}_{k'/k}(G')$ then all assertions in (ii) are satisfied for G as above, except for possibly the assertions concerning maximal k-tori and maximal k-split k-tori.

By [CGP, Cor. 9.4.13], we have the following results concerning tori in the basic non-reduced pseudo-simple case. The maximal k-split k-tori in G are maximal as k-tori (as is also the case over k' for the k'-group $G' \simeq \mathrm{Sp}_{2n}$), for each maximal k-torus T in G there is a unique maximal

k'-torus T' in $G' \simeq \mathrm{Sp}_{2n}$ such that $T \subseteq \xi_G^{-1}(\mathrm{R}_{k'/k}(T'))$, and for such T the map ξ_G carries T isomorphically onto the maximal k-torus in $\mathrm{R}_{k'/k}(T')$. In particular, $\dim T = \dim T'$ and T is k-split if and only if T' is k'-split, so the basic non-reduced pseudo-simple case is settled.

It remains to treat the case that G is basic exotic (with $p \in \{2, 3\}$). Since $[k : k^p] = p$, it follows from [CGP, Props. 7.3.1, 7.3.3, 7.3.5(1)] that there is a canonical k-homomorphism $f : G \twoheadrightarrow \overline{G}$ onto a connected semsimple k-group $\overline{G}$ that is absolutely simple and simply connected such that (a), (c), and (d) hold. The assertions in (b) are immediate from [CGP, Cor. 7.3.4]. ■

3. Preliminary simplifications

3.1 Smoothness

We now explain why the lack of a smoothness hypothesis on G in Theorem 1.3.1 involves no extra difficulty. This rests on the following useful lemma, which is [CGP, Lemma C.4.1].

Lemma 3.1.1. *Let X be a scheme locally of finite type over a field k. There is a unique geometrically reduced closed subscheme $X' \subseteq X$ such that $X'(K) = X(K)$ for all separable extension fields K/k. The formation of X' is functorial in X, and it commutes with the formation of products over k as well as separable extension of the ground field. In particular, if X is a k-group scheme then X' is a smooth k-subgroup scheme.*

Remark 3.1.2. Two consequences of Lemma 3.1.1 that will often be used without comment are that if G is a group scheme locally of finite type over a field k then (i) the maximal k-split k-tori in G are all $G(k)$-conjugate and (ii) for any maximal k-torus $T \subseteq G$ and extension field K/k, T_K is a maximal K-torus in G_K provided that G is k-smooth or K/k is separable. Lemma 3.1.1 reduces both assertions to the case of smooth G. Assertion (i) is [CGP, Prop. C.4.5] (via reduction to the smooth connected affine case, which is [CGP, Thm. C.2.3]). Assertion (ii) is [CGP, Lemma C.4.4].

Lemma 3.1.1 will be applied to separable extensions such as k_v/k for a global field k and place v of k. It is also used in the proof of the following result that will be needed later.

Proposition 3.1.3. *Let G be a group scheme locally of finite type over an arbitrary field k. Any smooth map $f : G \twoheadrightarrow G'$ onto a k-group G' locally of finite type carries maximal k-tori onto maximal k-tori, and likewise for maximal k-split k-tori. Moreover, every maximal k-torus* (*resp. maximal k-split k-torus*) *in G' lifts to one in G.*

Proof. This is [CGP, Prop. C.4.5(2)]. ■

To illustrate the usefulness of Lemma 3.1.1, we now reduce the proof of Theorem 1.3.1 to the case of smooth groups. Let k be a global field, G an affine k-group scheme of finite type, and G' as in Lemma 3.1.1 applied to G. The extension of fields k_v/k is separable for all places v of k, so the closed embedding $G'(k_v) \hookrightarrow G(k_v)$ of topological groups is an isomorphism for all v. By standard "spreading out" arguments there is a finite non-empty set S_0 of places of k (containing the archimedean places) such that the inclusion $G' \hookrightarrow G$ spreads out to a closed immersion of affine finite type $\mathscr{O}_{k,S_0}$-group schemes $G'_{S_0} \hookrightarrow G_{S_0}$. For any place $v \notin S_0$ we have $G_{S_0}(\mathscr{O}_v) \subseteq G(k_v) = G'(k_v) = G'_{S_0}(k_v)$, so $G_{S_0}(\mathscr{O}_v) = G'_{S_0}(\mathscr{O}_v)$ since $G'_{S_0}(\mathscr{O}_v) = G_{S_0}(\mathscr{O}_v) \cap G'(k_v)$ inside of $G'(k_v)$ (i.e., to check if an $\mathscr{O}_v$-valued solution to the $\mathscr{O}_{k,S_0}$-equations defining G_{S_0} satisfies the additional $\mathscr{O}_{k,S_0}$-equations defining G'_{S_0}, it is equivalent to work with the corresponding k_v-valued point). Hence, $G'(\mathbf{A}_k) = G(\mathbf{A}_k)$ as topological groups. The natural map $G'(k)\backslash G'(\mathbf{A}_k)/G'(k_S) \to$

$G(k)\backslash G(\mathbf{A}_k)/G(k_S)$ is therefore a homeomorphism for all S, so one side is quasi-compact if and only if the other side is. Thus, G has finite class numbers provided that G' does, so to prove Theorem 1.3.1 for G it suffices to prove it for the smooth G'. Note that G' may be disconnected even if G is connected (e.g., see [CGP, Rem. C.4.2]).

Since non-affine groups will arise in our considerations in §7.5, it is convenient to record two general structure theorems for smooth connected groups over a field. The first is well-known, but only applicable over perfect fields, whereas the second is not widely known but has been available for a long time and is very useful over imperfect fields.

THEOREM 3.1.4. (Chevalley) *Let G be a smooth connected group over a perfect field k. There is a unique short exact sequence of smooth connected k-groups*

$$1 \to H \to G \to A \to 1$$

with H affine and A an abelian variety.

Proof. Chevalley's original proof is given in [Ch], but it may be difficult to read nowadays due to the style of algebraic geometry that is used. See [C1] for a modern exposition. ■

If the perfectness hypothesis on k is dropped in Theorem 3.1.4 then the conclusion can fail; counterexamples are given in [CGP, Ex. A.3.8] over every imperfect field. Here is a remarkable substitute for Theorem 3.1.4 that is applicable over all fields (and whose proof uses Theorem 3.1.4 over an algebraic closure of the ground field):

THEOREM 3.1.5. *Let F be a field and G a smooth connected F-group. The F-algebra $\mathscr{O}(G)$ is finitely generated and smooth, and when $G^{\mathrm{aff}} := \mathrm{Spec}(\mathscr{O}(G))$ is endowed with its natural F-group structure the natural map $G \to G^{\mathrm{aff}}$ is a surjection with smooth connected central kernel Z satisfying $\mathscr{O}(Z) = F$. If $\mathrm{char}(F) > 0$ then Z is semi-abelian (i.e., an extension of an abelian variety by an F-torus).*

The centrality of Z makes this extension structure on G very convenient for cohomological arguments (in contrast with Theorem 3.1.4, where the commutative term is the quotient).

Proof. See [DG, III, §3, 8.2, 8.3] for all but the semi-abelian property in nonzero characteristic. This special feature in nonzero characteristic is proved in [Bri, Prop. 2.2] resting on the commutative case of Theorem 3.1.4 over $\overline{F}$ (and was independently proved in [SS] by another method). A proof of the semi-abelian property is also given in [CGP, Thm. A.3.9]. ■

3.2 Connectedness

We now review (in scheme-theoretic language) an argument of Borel [Bo1, 1.9] to show that G has finite class numbers if G^0 has finite class numbers, where G is an affine group scheme of finite type over a global field k and G^0 is its identity component. Since G^0 is a closed normal subgroup subscheme of G [SGA3, $\mathrm{IV_A}$, 2.3], $G^0(\mathbf{A}_k)$ is a closed normal subgroup of $G(\mathbf{A}_k)$. In particular, the quotient space $G(\mathbf{A}_k)/G^0(\mathbf{A}_k)$ is locally compact and Hausdorff, and it is naturally a topological group. By standard "spreading out" arguments, for a suitable finite non-empty set S of places of k (containing the archimedean places) there exists an affine group scheme G_S of finite type over $\mathrm{Spec}\,\mathscr{O}_{k,S}$ with generic fiber G and an open and closed subgroup G_S^0 of G_S that fiberwise coincides with the identity component of the fibers of G_S over $\mathrm{Spec}\,\mathscr{O}_{k,S}$. This interpolation of the fibral identity components is used in the proof of the next result.

Proposition 3.2.1. (Borel) *For any global field k and affine k-group scheme G of finite type, the Hausdorff quotient $G(\mathbf{A}_k)/G^0(\mathbf{A}_k)$ is compact. In fact, it is profinite.*

Proof. Let $G' \subseteq G$ be as in Lemma 3.1.1. As we have seen in the discussion following Proposition 3.1.3, $G'(\mathbf{A}_k) = G(\mathbf{A}_k)$ as topological groups. Likewise, $(G')^0(\mathbf{A}_k) \subseteq G^0(\mathbf{A}_k)$ since $(G')^0 \subseteq G^0$, so $G(\mathbf{A}_k)/G^0(\mathbf{A}_k)$ is topologically a Hausdorff quotient group of $G'(\mathbf{A}_k)/(G')^0(\mathbf{A}_k)$. We may therefore replace G with G' so as to assume that G is smooth. The smooth case was treated by Borel using the crutch of GL_n. A well-known expert in algebraic groups requested an exposition of Borel's argument without that crutch; this is given in Appendix C, using G_S and G_S^0 as mentioned above. ■

To get a feeling for Proposition 3.2.1 consider the special case when G is the constant k-group associated to a finite group Γ. In this case G^0 is trivial and $G(\mathbf{A}_k)$ is the set of Γ-tuples of mutually orthogonal idempotents in $\mathbf{A}_k$ with sum adding up to 1. In other words, if V_k denotes the set of places of k (index set for the "factors" of $\mathbf{A}_k$), then $G(\mathbf{A}_k)$ is the set $\mathrm{Hom}_{\mathrm{Set}}(V_k, \Gamma) = \prod_{V_k} \Gamma$ (product with index set V_k). The topology induced by $\mathbf{A}_k$ is equal to the product topology, so profiniteness is evident in this case.

Corollary 3.2.2. (Borel) *An affine group scheme G of finite type over a global field k has finite class numbers if its identity component G^0 does.*

Proof. The inclusion

$$G(k)/G^0(k) \hookrightarrow (G/G^0)(k)$$

implies that $G(k)/G^0(k)$ is finite (since G/G^0 is k-finite). Let S be a finite non-empty set of places of k containing S_∞ and let K be a compact open subgroup in $G(\mathbf{A}_k^S)$, so $K^0 := K \cap G^0(\mathbf{A}_k^S)$ is a compact open subgroup of $G^0(\mathbf{A}_k^S)$ (since $G^0(\mathbf{A}_k^S)$ is a closed subgroup of $G(\mathbf{A}_k^S)$). By the hypothesis that G^0 has finite class numbers with respect to S, there exists a finite set $\{\gamma_j^0\}$ in $G^0(\mathbf{A}_k^S)$ such that

$$G^0(\mathbf{A}_k^S) = \coprod G^0(k)\gamma_j^0 K^0.$$

By Proposition 3.2.1, $G(\mathbf{A}_k^S)/G^0(\mathbf{A}_k^S)$ is compact, so there exists a finite subset $\{g_i\}$ in $G(\mathbf{A}_k^S)$ such that

$$G(\mathbf{A}_k^S) = \coprod G^0(\mathbf{A}_k^S) g_i K = \coprod G^0(k)\gamma_j^0 K^0 g_i K.$$

Since $G^0(k) \subseteq G(k)$ and each compact open subset $K^0 g_i K$ in $G(\mathbf{A}_k^S)$ is a finite union of right cosets $g_{i,\alpha}K$, we obtain finiteness of $G(k)\backslash G(\mathbf{A}_k^S)/K$. ■

4. Finiteness properties of tori and adelic quotients

This section largely consists of well-known facts (for which we include some proofs, as a convenience to the reader). We gather them here for ease of reference, and incorporate generalizations (e.g., removal of smoothness hypotheses) that will be needed later. The only new result in this section is Proposition 4.1.9.

4.1 Tori

Let L be a (possibly archimedean) local field and let $|\cdot|_L$ be its normalized absolute value. For an arbitrary torus T over L, we define

$$T(L)^1 = \bigcap_{\chi \in \mathrm{X}_L(T)} \ker |\chi|_L.$$

For example, $T(L)^1 = T(L)$ if T is L-anisotropic. The subgroup $T(L)^1 \subseteq T(L)$ is functorial in T, its formation commutes with direct products in T, and it contains all compact subgroups of $T(L)$. The first two lemmas below are special cases of [La, Prop. 1.2(ii)] and [La, Lemma 1.3(ii)] respectively.

Lemma 4.1.1. *For a local field L, the maximal compact subgroup of $T(L)$ is $T(L)^1$.*

Proof. The problem is to prove that $T(L)^1$ is compact. By functoriality with respect to the closed immersion of L-tori

$$T \hookrightarrow \mathrm{R}_{L'/L}(T_{L'})$$

for a finite separable extension L'/L that splits T, it is enough to consider the special case $T = \mathrm{R}_{L'/L}(\mathrm{GL}_1)$. In this case $T(L) = {L'}^{\times}$ topologically and $\mathrm{X}_L(T)$ is infinite cyclic with $\mathrm{N}_{L'/L}$ as a nontrivial element, so

$$T(L)^1 = \ker(T(L) \overset{\mathrm{N}_{L'/L}}{\to} \mathrm{GL}_1(L) = L^{\times} \overset{|\cdot|_L}{\to} \mathbf{R}^{\times}_{>0}) = \mathscr{O}^{\times}_{L'}.$$

■

Lemma 4.1.2. *Let G be a smooth group scheme over a local field L, and T an L-torus.*

(i) *Let $G \twoheadrightarrow T$ be a smooth surjective L-homomorphism. The natural map $G(L) \to T(L)$ has open image with finite index.*

(ii) *If $T' \to T$ is a map between L-tori and its restriction $T'_0 \to T_0$ between maximal L-split subtori is surjective then the induced map $T'(L)/T'(L)^1 \to T(L)/T(L)^1$ modulo maximal compact subgroups has image with finite index.*

Proof. We first reduce (i) to (ii). Since $G \to T$ is smooth, $G(L)$ has open image in $T(L)$ and hence (by Lemma 4.1.1) has image with finite index if and only if the image of $G(L)$ in $T(L)/T(L)^1$ has finite index. By Proposition 3.1.3, any maximal L-torus T' in G maps onto T. Thus, the maximal L-split subtorus in T' maps onto that of T, so it suffices to prove (ii).

The map $T_0(L)/T_0(L)^1 \to T(L)/T(L)^1$ is obviously injective, and we claim that its cokernel is finite. There is an isogeny $\pi : T_0 \times T_1 \to T$ with $T_1 \subseteq T$ the maximal L-anisotropic subtorus, so $T_1(L)$ is compact and therefore lies in $T(L)^1$. Hence, $T_0(L) \to T(L)/T(L)^1$ has cokernel that is a subquotient of the group $\mathrm{H}^1(L, \ker \pi)$ that is finite when $\mathrm{char}(L) = 0$.

Now assume $\mathrm{char}(L) > 0$, or more generally that L is non-archimedean. Thus, $T(L)^1$ is open in $T(L)$ and so its image in the compact quotient $T(L)/T_0(L) = (T/T_0)(L)$ has finite index. By applying the same reasoning to T' in the role of T, the map $T'_0(L)/T'_0(L)^1 \to T'(L)/T'(L)^1$ is injective with finite cokernel. Hence, we may and do assume that T and T' are L-split.

Consider the canonical isomorphism $T(L)/T(L)^1 \simeq \mathrm{X}_{*,L}(T) := \mathrm{Hom}_L(\mathrm{GL}_1, T)$ defined by $\lambda \mapsto \lambda(\pi) \bmod T(L)^1$ for any uniformizer π of $\mathscr{O}_L$ (the choice of which does not matter). The map $\mathrm{X}_{*,L}(T') \to \mathrm{X}_{*,L}(T)$ has image with finite index, since T and T' are L-split and surjections between L-tori admits sections in the isogeny category of L-tori. Hence, the map $T'(L)/T'(L)^1 \to T(L)/T(L)^1$ has image with finite index. ■

Lemma 4.1.3. *Let L be a field, L' a nonzero finite reduced L-algebra, G' an L'-group scheme of finite type, and $G := \mathrm{R}_{L'/L}(G')$ the Weil restriction to L. For any maximal L'-split torus $T' \subseteq G'$, the maximal L-split torus T in $\mathrm{R}_{L'/L}(T')$ is a maximal L-split torus in G. Moreover, $T' \mapsto T$ is a bijection between sets of maximal split tori. The same holds for the set of maximal tori.*

In particular, if L is a non-archimedean local field and L'/L is a finite extension field then for any such pair (T, T') of maximal split tori the subgroup of $T'(L')$ generated by $T(L) \subseteq G(L) = G'(L')$ and any compact open subgroup of $T'(L')$ has finite index in $T'(L')$.

Proof. The final part follows from the rest by Lemma 4.1.2(ii). By (the proof of) Lemma 3.1.1 we can assume that G' is L'-smooth. In the smooth affine case, this is [CGP, Prop. A.5.15(2)]. Using Proposition 3.1.3 and [CGP, Lemma C.4.4], the proof in the affine case works in general. ■

The interesting case of Lemma 4.1.3 is when T' has a nontrivial fiber over a factor field of L' that is not separable over L, as then $\mathrm{R}_{L'/L}(T')$ is not an L-torus. We only need the lemma for smooth affine G'. For the reader interested in the general case, note that $\mathrm{R}_{L'/L}(G')$ makes sense as an L-scheme because G' is quasi-projective [CGP, Prop. A.3.5].

Lemma 4.1.4. *Let U be a k-split smooth connected unipotent group over a field k, and let T be a k-torus. Any extension E of U by T is split.*

Proof. This is [SGA3, XIV, 6.1.A(ii)], but for the convenience of the reader we give a direct argument here. Since E is smooth and connected, such an extension must be central (as the automorphism scheme $\underline{\mathrm{Aut}}(T)$ is étale). If a splitting exists then it is unique (since $\mathrm{Hom}_k(U, T) = 1$), so we can assume k is separably closed and thus T is k-split. We may therefore assume $T = \mathrm{GL}_1$. Also, by uniqueness of the splitting we can use a composition series for the k-split U to reduce to the case $U = \mathbf{G}_{\mathrm{a}}$. Since $\mathrm{Pic}(\mathbf{G}_{\mathrm{a}}) = 1$, the quotient map $E \twoheadrightarrow \mathbf{G}_{\mathrm{a}}$ has a k-scheme section, and we can arrange that it respects the identity points. Thus, $E = \mathrm{GL}_1 \times \mathbf{G}_{\mathrm{a}}$ as k-schemes such that the identity is $(1, 0)$ and the group law is $(t, x)(t', x') = (tt' \cdot f(x, x'), x + x')$ for some map of k-schemes $f : \mathbf{G}_{\mathrm{a}} \times \mathbf{G}_{\mathrm{a}} \to \mathrm{GL}_1$ satisfying $f(0, 0) = 1$. The only such f is the constant map $f = 1$. ■

Proposition 4.1.5. *Let G be a smooth connected affine group over a local field L and let $T \subseteq G$ be a maximal L-split torus. Assume that G is either commutative with no L-subgroup isomorphic to $\mathbf{G}_{\mathrm{a}}$ or is in one the following classes of L-groups: semisimple, basic exotic pseudo-reductive (with $\mathrm{char}(L) \in \{2, 3\}$), or basic non-reduced pseudo-simple (with $\mathrm{char}(L) = 2$).*

An open subgroup $\mathscr{U} \subseteq G(L)$ has finite index in $G(L)$ if and only if $\mathscr{U} \cap T(L)$ has finite index in $T(L)$.

See Proposition 4.1.9 for a generalization, building on the cases considered here.

Proof. The "only if" direction is obvious, so we focus on the converse. The case of archimedean L is trivial, since it is well-known that the topological identity component $G(L)^0$ has finite index in $G(L)$ for archimedean L. Hence, we can assume L is non-archimedean. First we treat the case of commutative G containing no $\mathbf{G}_{\mathrm{a}}$. Note that G/T contains no GL_1, by maximality of T. The L-group G/T also cannot contain $\mathbf{G}_{\mathrm{a}}$ as an L-subgroup, due to Lemma 4.1.4 applied to the preimage of such a $\mathbf{G}_{\mathrm{a}}$ in G. Thus, G/T contains neither GL_1 nor $\mathbf{G}_{\mathrm{a}}$ as an L-subgroup.

We claim that $(G/T)(L)$ is compact. Granting this, let us show how to conclude the commutative case. Since T is L-split, we know that $G(L)/T(L) = (G/T)(L)$ topologically. Hence, $G(L)/T(L)$ is compact, so any open subgroup of $G(L)$ has finite-index image in $G(L)/T(L)$ for topological reasons. Any open subgroup of $G(L)$ that meets $T(L)$ in a finite-index subgroup of $T(L)$ therefore has finite index in $G(L)$, so we would be done in the commutative case.

By replacing G with G/T, we have reduced the commutative case to G that do not contain GL_1 or $\mathbf{G}_{\mathrm{a}}$ as L-subgroups. Let T' be a maximal L-torus in G, so $T'(L)$ is compact (Lemma 4.1.1) and $G(L)/T'(L)$ is an open subgroup of $(G/T')(L)$. The group G/T' is smooth, connected, and unipotent, so it suffices to show that $(G/T')(L)$ is compact. By [Oes, VI, §1], it is equivalent to show that G/T' does not contain $\mathbf{G}_{\mathrm{a}}$ as an L-subgroup. This is another application of Lemma 4.1.4 since G is assumed to not contain $\mathbf{G}_{\mathrm{a}}$ as an L-subgroup.

Next we consider the case when G is semisimple. This case is a well-known result of Tits, and for the convenience of the reader we now recall the argument. Let $G(L)^+$ be the normal subgroup in $G(L)$ generated by the L-rational points of the unipotent radicals of the minimal parabolic L-subgroups of G. Since G is semisimple, by [BoT2, 6.2, 6.14] the group $G(L)^+$ is a closed subgroup in $G(L)$ and the quotient space $G(L)/G(L)^+$ is compact. Thus, the open subgroup $\mathscr{U} G(L)^+/G(L)^+$ is also compact. The natural bijective continuous homomorphism $\mathscr{U}/(\mathscr{U} \cap G(L)^+) \to \mathscr{U} G(L)^+/G(L)^+$ is open and hence a homeomorphism, so $\mathscr{U}/(\mathscr{U} \cap G(L)^+)$ is compact. If $\mathscr{U} \cap G(L)^+$ is also compact then it follows that $\mathscr{U}$ is compact, so $\mathscr{U} \cap T(L)$ is compact. This would force $T(L)$ to be compact since $\mathscr{U} \cap T(L)$ is a subgroup of finite index in $T(L)$ by hypothesis, so $T = 1$ since T is an L-split torus. That is, if $\mathscr{U} \cap G(L)^+$ is compact then the semisimple L-group G is L-anisotropic, in which case $G(L)$ is compact (see [Pr2]) and so the open subgroup $\mathscr{U}$ trivially has finite index.

Thus, we can assume that $\mathscr{U} \cap G(L)^+$ is non-compact. It is a theorem of Tits (proved in [Pr2]) that every proper open subgroup of $G(L)^+$ is compact, so $\mathscr{U} \cap G(L)^+ = G(L)^+$. That is, $\mathscr{U}$ contains $G(L)^+$. The quotient $\mathscr{U}/G(L)^+$ is an open subgroup in the compact group $G(L)/G(L)^+$, so it has finite index and hence $\mathscr{U}$ has finite index in $G(L)$.

Finally, suppose $\mathrm{char}(L) \in \{2, 3\}$ and G is either basic exotic pseudo-reductive or basic non-reduced pseudo-simple (with $\mathrm{char}(L) = 2$). Using the quotient map $f : G \to \overline{G}$ provided by Theorem 2.3.8(ii), by Lemma 4.1.2(ii) the problem for G reduces to the analogue for $\overline{G}$. (The key point with Lemma 4.1.2(ii) is that it enables us to bypass the fact that a non-étale isogeny between L-split L-tori never has finite-index image on L-points.) In the basic exotic case the L-group $\overline{G}$ is semisimple (even simply connected), and this was handled above. In the basic non-reduced case we have $\overline{G} \simeq \mathrm{R}_{L'/L}(\overline{G}')$ for $L' = L^{1/2}$ and $\overline{G}' \simeq \mathrm{Sp}_{2n}$ as L'-groups, so naturally $\overline{G}(L) \simeq \overline{G}'(L')$ as topological groups. An application of Lemma 4.1.3 then handles the interaction of rational points of tori under this topological group isomorphism, reducing the problem for $\overline{G}$ over L to the settled case of $\overline{G}'$ over L'. ■

We next record some standard cohomological finiteness properties of group schemes of multiplicative type over non-archimedean local fields, especially to allow non-smooth groups over local function fields. First we recall Shapiro's Lemma, stated in a form that allows inseparable field extensions (as we will require later).

Lemma 4.1.6. *Let k be a field, k' a nonzero finite reduced k-algebra, and $\{k'_i\}$ its set of factor fields. Let G' be a smooth affine k'-group, and G'_i its k'_i-fiber.*

There is a natural isomorphism of pointed sets

$$\mathrm{H}^1(k, \mathrm{R}_{k'/k}(G')) \simeq \mathrm{H}^1(k', G') = \prod \mathrm{H}^1(k'_i, G'_i),$$

and if G' is commutative then this is an isomorphism of groups. Moreover, in the commutative *case there are natural group isomorphisms*

$$\mathrm{H}^m(k, \mathrm{R}_{k'/k}(G')) \simeq \mathrm{H}^m(k', G') = \prod \mathrm{H}^m(k'_i, G'_i)$$

for all $m \geqslant 1$.

Proof. This is [Oes, IV, 2.3] since $\mathrm{R}_{k'/k}(G') = \prod \mathrm{R}_{k'_i/k}(G'_i)$. ∎

Proposition 4.1.7. *Let k be a non-archimedean local field.*

(i) *If T is a k-torus then $\mathrm{H}^1(k,T)$ is finite.*

(ii) *If M is a finite k-group scheme of multiplicative type then $\mathrm{H}^2(k,M)$ is finite.*

Proof. For a k-torus T, consider the pairing

$$\mathrm{H}^1(k,T) \times \mathrm{H}^1(k, \mathrm{X}(T)) \to \mathrm{H}^2(k, \mathrm{GL}_1) = \mathbf{Q}/\mathbf{Z},$$

where $\mathrm{X}(T) := \mathrm{Hom}_{k_s}(T_{k_s}, \mathrm{GL}_1)$ is the geometric character group (for a separable closure k_s/k). Since $\mathrm{X}(T)$ is a finite free $\mathbf{Z}$-module, it follows from local class field theory (see [Mi2, I, Thm. 1.8(a)]) that this pairing identifies $\mathrm{H}^1(k,T)$ with the $\mathbf{Q}/\mathbf{Z}$-dual of $\mathrm{H}^1(k,\mathrm{X}(T))$. Thus, for (i) we just have to show that $\mathrm{H}^1(k,\mathrm{X}(T))$ is finite, and this follows by using inflation-restriction with respect to a finite Galois extension k'/k that splits the discrete torsion-free $\mathrm{Gal}(k_s/k)$-module $\mathrm{X}(T)$.

Now consider M as in (ii). Let F/k be a finite Galois splitting field for the finite étale Cartier dual of M, with Galois group $\Gamma = \mathrm{Gal}(F/k)$. This Cartier dual is a quotient of a power of $\mathbf{Z}[\Gamma]$ as a Γ-module, so M is naturally a k-subgroup of a k-torus T that is a power of $\mathrm{R}_{F/k}(\mathrm{GL}_1)$. The exact sequence

$$1 \to M \to T \to \mathscr{T} \to 1$$

with $\mathscr{T} := T/M$ a k-torus gives an exact sequence

$$\mathrm{H}^1(k,\mathscr{T}) \to \mathrm{H}^2(k,M) \to \mathrm{H}^2(k,T)[n]$$

where n is the order of M. Since $\mathrm{H}^1(k,\mathscr{T})$ is finite, it suffices to prove that $\mathrm{H}^2(k,T)[n]$ is finite for any integer $n \geqslant 1$. By Lemma 4.1.6, $\mathrm{H}^2(k,T)$ is a power of $\mathrm{Br}(F)$, and $\mathrm{Br}(F)[n]$ is finite by local class field theory. ∎

For later use, we require a generalization of Proposition 4.1.5 that rests on the structure theory in §2.3 in the local function field case. First, we introduce a concept that arose in [BoT3, §6], using the terminology given for it later in [BTII, 1.1.12].

Definition 4.1.8. A group scheme H over a field F is *quasi-reductive* if it is smooth, affine, and contains no nontrivial F-split smooth connected unipotent normal F-subgroup.

A smooth connected unipotent normal F-subgroup V in a quasi-reductive F-group H cannot contain $\mathbf{G}_{\mathrm{a}}$ as an F-subgroup. Indeed, if U_0 is such an F-subgroup of V then the $H(F_s)$-conjugates of $(U_0)_{F_s}$ generate a nontrivial smooth connected normal F_s-subgroup U_s of H_{F_s} that descends to an F-subgroup $U \subseteq V$ (so it is unipotent) and by construction admits no quotient that is F_s-wound in the sense of Definition 7.1.1. Thus, U_s is F_s-split (by [CGP, Thm. B.3.4] applied over F_s), so U is F-split (by [CGP, Thm. B.3.4] applied over F). But $U \neq 1$, so this contradicts that H is quasi-reductive over F. (It follows that quasi-reductivity is equivalent to the condition that $\mathscr{R}_{u,F}(H)$ is F-wound in the sense of Definition 7.1.1ff.)

Proposition 4.1.9. *Let L be a local field and H a smooth affine L-group that is quasi-reductive in the sense of Definition 4.1.8. Let $T_0 \subseteq H$ be a maximal L-split L-torus. An open subgroup $\mathscr{U} \subseteq H(L)$ has finite index if and only if $\mathscr{U} \cap T_0(L)$ has finite index in $T_0(L)$.*

Proof. We focus on the nontrivial implication "$\Leftarrow$". The archimedean case is trivial, so we can assume that L is non-archimedean. If H is commutative then the commutative case of Proposition 4.1.5 implies that $\mathscr{U}$ has finite index in $H(L)$. Hence, we can assume that H is not commutative. We will first treat the case of pseudo-reductive H, and then use this to handle the general quasi-reductive case.

With H now assumed to be pseudo-reductive, by Theorem 2.3.6(ii) (in case $\mathrm{char}(L) = 2$) and Theorem 2.3.8 we may and do assume H is a non-commutative generalized standard pseudo-reductive L-group. (This reduction step uses Lemma 4.1.2(ii) and Lemma 4.1.3, exactly as in the treatment of basic non-reduced cases at the end of the proof of Proposition 4.1.5.)

Choose a maximal L-torus $T \subseteq H$ containing T_0, and let $C = Z_H(T)$ be the corresponding Cartan k-subgroup of H. Consider the generalized standard presentation $(H', k'/k, T', C)$ of H adapted to T; see Definition 2.3.3 and Remark 2.3.4. In particular, there is a factorization diagram

$$\mathrm{R}_{L'/L}(C') \to C \to \mathrm{R}_{L'/L}(C'/Z_{H'})$$

such that

$$H \simeq (\mathrm{R}_{L'/L}(H') \rtimes C)/\,\mathrm{R}_{L'/L}(C'). \tag{4.1.1}$$

Note that T_0 is the maximal L-split torus in C, and C does not contain $\mathbf{G}_\mathrm{a}$ as an L-subgroup (since C is pseudo-reductive over L). Thus, by the commutative case of Proposition 4.1.5, $\mathscr{U} \cap C(L)$ has finite index in $C(L)$.

Write $L' \simeq \prod L'_i$ as a finite product of local fields of finite degree (but possibly not separable) over L. Let H'_i denote the fiber of H' over the factor field L'_i of L', so either H'_i is a simply connected and absolutely simple semisimple L'_i-group or $\mathrm{char}(L) \in \{2, 3\}$ and H'_i is a basic exotic pseudo-reductive L'_i-group. Let C'_i denote the L'_i-fiber of C', so it is a Cartan L'_i-subgroup of H'_i. In particular, C'_i is a torus when H'_i is semisimple. Suppose instead that H'_i is basic exotic, so the quotient map $H'_i \twoheadrightarrow \overline{H}'_i$ provided by Theorem 2.3.8(ii) carries C'_i onto a Cartan L'_i-subgroup $\overline{C}'_i$ in $\overline{H}'_i$. For a separable closure $L'_{i,s}$ of L'_i, the bijectivity of $H'_i(L'_{i,s}) \to \overline{H}'_i(L'_{i,s})$ implies that the injective map $C'_i(L'_{i,s}) \to \overline{C}'_i(L'_{i,s})$ is surjective (because C'_i is its own centralizer in H'_i). Hence, $\mathrm{H}^m(L'_i, C'_i) \to \mathrm{H}^m(L'_i, \overline{C}'_i)$ is an isomorphism for all m in such cases, with $\overline{C}'_i$ a torus since $\overline{H}'_i$ is semisimple.

By Lemma 4.1.6 and Proposition 4.1.7 (applied over the factor fields L'_i), it follows that $\mathrm{H}^1(L, \mathrm{R}_{L'/L}(C'))$ is always finite. Thus, the *central* pushout presentation (4.1.1) implies that the open map

$$\mathrm{R}_{L'/L}(H')(L) \rtimes C(L) \to H(L) \tag{4.1.2}$$

has normal image $\mathscr{V}$ with finite index. It therefore suffices to show that $\mathscr{U} \cap \mathscr{V}$ has finite index in $\mathscr{V}$.

We have just seen that $\mathscr{U}$ meets the image of $C(L) \hookrightarrow H(L)$ with finite index in $C(L)$, so the image of $\mathscr{U} \cap \mathscr{V}$ in the quotient $\mathscr{V}''$ of $\mathscr{V}$ modulo the normal image of $\mathrm{R}_{L'/L}(H')(L)$ has finite index. It is trivial to check that if $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ is an exact sequence of abstract groups then a subgroup of Γ has finite index if (and only if) its image in Γ'' has finite index in Γ'' and its intersection with Γ' has finite index in Γ'. Thus, it remains to check that the open preimage of $\mathscr{U} \cap \mathscr{V}$ (equivalently, of $\mathscr{U}$) under (4.1.2) meets $\mathrm{R}_{L'/L}(H')(L)$ in a subgroup of $\mathrm{R}_{L'/L}(H')(L)$ with finite index.

By [CGP, Thm. C.2.3], any two maximal split tori in a smooth connected affine group over a field are conjugate by a rational point. Applying this to H and using the *functoriality* of $(H', L'/L)$ with respect to L-automorphisms of H, T_0 contains the image of a maximal L-split torus $\overline{T}_0$ in $\mathrm{R}_{L'/L}(H')$. The open preimage $\overline{\mathscr{U}}$ of $\mathscr{U}$ in $\mathrm{R}_{L'/L}(H')(L)$ therefore meets $\overline{T}_0(L)$ in a finite-index subgroup. Thus, we just need to prove the analogue of Proposition 4.1.5 for the L-group $\mathrm{R}_{L'/L}(H')$.

The maximal L-split tori in

$$\mathrm{R}_{L'/L}(H') \simeq \prod_i \mathrm{R}_{L'_i/L}(H'_i)$$

are products of maximal L-split tori in the factors. Applying Lemma 4.1.3 to each factor therefore gives that $\overline{T}_0 = \prod \overline{T}_{0,i}$, with $\overline{T}_{0,i}$ the maximal L-split torus in $\mathrm{R}_{L'_i/L}(T'_{0,i})$ for some maximal L'_i-split torus $T'_{0,i}$ in H'_i. Thus, by the final part of Lemma 4.1.3, the open subgroup $\overline{\mathscr{U}}$ viewed in $\prod_i H'_i(L'_i)$ meets $\prod T'_{0,i}(L'_i)$ in a finite-index subgroup. The technique of proof of Proposition 4.1.5 in the semisimple and basic exotic cases applies to open subgroups of the product $\prod_i H'_i(L'_i)$ since each H'_i is either connected semisimple or basic exotic over L'_i with maximal L'_i-split torus $T'_{0,i}$ for all i. This settles the general case of pseudo-reductive H.

Now consider any quasi-reductive L-group H. In characteristic 0 such H are reductive, so we can apply the pseudo-reductive case to H^0. Thus, we may assume $\mathrm{char}(L) = p > 0$. We may also assume H is connected, and we let $U \subseteq H$ be the maximal smooth connected unipotent normal L-subgroup, so H/U is pseudo-reductive over L.

Since $H \to H/U$ is a smooth surjection with unipotent kernel, the map $H(L) \to (H/U)(L)$ is open and T_0 is carried isomorphically onto a maximal L-split torus in H/U. The argument following Definition 4.1.8 shows that the smooth normal L-subgroup U does not contain $\mathbf{G}_{\mathrm{a}}$ as an L-subgroup, since H is quasi-reductive. By [Oes, VI, §1] it follows that the group $U(L)$ is compact. Thus, $\mathscr{U} \cap U(L)$ has finite index in $U(L)$, so we can replace $\mathscr{U}$ with the open subgroup $\mathscr{U} \cdot U(L)$ in which $\mathscr{U}$ has finite index in order to reduce to the case $U(L) \subseteq \mathscr{U}$. The settled pseudo-reductive case can be applied to the open subgroup $\mathscr{U}/U(L) \subseteq (H/U)(L)$ and the L-torus T_0 viewed as a maximal L-split torus in H/U, so $\mathscr{U}/U(L)$ has finite index in $(H/U)(L)$ and hence in $H(L)/U(L)$. This proves that $\mathscr{U}$ has finite index in $H(L)$. ■

4.2 Adelic quotients

Throughout this section, k is a global field. We begin by recalling a useful general result in the theory of topological groups.

Theorem 4.2.1. *Let G be a second-countable locally compact Hausdorff topological group, and X a locally compact Hausdorff topological space endowed with a continuous right G-action. Let $x \in X$ be a point and let $G_x \subseteq G$ be its stabilizer for the G-action. If the orbit $x \cdot G$ is locally closed in X then the natural map $G_x \backslash G \to X$ induced by $g \mapsto xg$ is a homeomorphism onto the orbit of x.*

Proof. See [Bou, IX, §5] for a proof in a more general setting. The role of second-countability is so that the Baire category theorem may be applied. ■

Definition 4.2.2. For an affine k-group scheme H of finite type and a k-rational character

$\chi \in \mathrm{X}_k(H) := \mathrm{Hom}_k(H, \mathrm{GL}_1)$, let

$$|\chi| : H(\mathbf{A}_k) \to \mathbf{R}^\times_{>0}$$

denote the continuous composition of $\chi : H(\mathbf{A}_k) \to \mathrm{GL}_1(\mathbf{A}_k) = \mathbf{A}_k^\times$ and the idelic norm homomorphism $\|\cdot\|_k : \mathbf{A}_k^\times \to \mathbf{R}^\times_{>0}$. The closed subgroup $H(\mathbf{A}_k)^1 \subseteq H(\mathbf{A}_k)$ is defined to be

$$H(\mathbf{A}_k)^1 := \bigcap_{\chi \in \mathrm{X}_k(H)} \ker |\chi|.$$

Example 4.2.3. If H is a (connected) semisimple k-group, a unipotent k-group, an anisotropic k-torus, or more generally $\mathrm{X}_k(H) = \{1\}$, then $H(\mathbf{A}_k)^1 = H(\mathbf{A}_k)$. In general, the subgroup $H(\mathbf{A}_k)^1 \subseteq H(\mathbf{A}_k)$ is normal and functorial in H, and $H(\mathbf{A}_k)/H(\mathbf{A}_k)^1$ is commutative. If k is a global function field then $H(\mathbf{A}_k)^1$ is open in $H(\mathbf{A}_k)$ because the idelic norm is discretely-valued for such k and $\mathrm{X}_k(H)$ is finitely generated over $\mathbf{Z}$.

Lemma 4.2.4. *Let $f : T' \to T$ be a k-homomorphism between k-tori such that f restricts to an isogeny between maximal k-split subtori. The induced map $T'(\mathbf{A}_k)/T'(\mathbf{A}_k)^1 \to T(\mathbf{A}_k)/T(\mathbf{A}_k)^1$ is an isomorphism in the number field case and is injective with finite-index image in the function field case.*

Proof. When T' and T are k-split, so f is an isogeny, we can choose compatible bases of the character groups to reduce to the trivial case when $T' = T = \mathrm{GL}_1$ and f is the nth-power map for a nonzero integer n. In general, the hypotheses imply that f induces an isogeny between maximal k-split quotients. Hence, it suffices to treat the case when T is the maximal k-split quotient T'_0 of T'. Every k-rational character of T' factors through T'_0, so injectivity always holds. Since T' contains a k-split subtorus S such that $S \to T'_0$ is a k-isogeny, the settled split case applied to this isogeny settles the general case. ■

Our interest in Definition 4.2.2 is due to the following lemma (which is well-known in the smooth case, and will be useful in the non-smooth case in Appendix A):

Lemma 4.2.5. *Let H be a closed k-subgroup scheme of an affine k-group scheme H' of finite type. The natural map of coset spaces*

$$H(k)\backslash H(\mathbf{A}_k)^1 \to H'(k)\backslash H'(\mathbf{A}_k)^1$$

is a closed embedding. In particular, the map $H(k)\backslash H(\mathbf{A}_k)^1 \to H'(k)\backslash H'(\mathbf{A}_k)$ is a closed embedding.

Proof. The target is a locally compact Hausdorff space admitting a continuous right action by $H'(\mathbf{A}_k)^1$ and hence by $H(\mathbf{A}_k)^1$, and $H(\mathbf{A}_k)^1$ is a second-countable locally compact Hausdorff group. It follows from Theorem 4.2.1 that for $x \in H'(k)\backslash H'(\mathbf{A}_k)^1$ and its stabilizer subgroup S_x in $H(\mathbf{A}_k)^1$, the natural orbit map

$$S_x\backslash H(\mathbf{A}_k)^1 \to H'(k)\backslash H'(\mathbf{A}_k)^1$$

is a homeomorphism onto the $H(\mathbf{A}_k)^1$-orbit of x if the orbit is closed. Taking x to be the coset of the identity gives $S_x = H'(k) \cap H(\mathbf{A}_k)^1 = H(k)$, and so we are reduced to proving that the $H(\mathbf{A}_k)^1$-orbit of the identity coset in $H'(k)\backslash H'(\mathbf{A}_k)^1$ is closed.

We have to prove that $H'(k)H(\mathbf{A}_k)^1$ is closed in $H'(\mathbf{A}_k)^1$. An elegant proof is given in [Oes, IV, 1.1], where it is assumed that H' and H are smooth. This smoothness is not needed. More precisely, the only role of smoothness is to invoke the standard result that if G is a smooth affine group scheme over a field k and G' is a smooth closed subgroup scheme then there is a closed immersion of k-groups $G \hookrightarrow \mathrm{GL}(V)$ for a finite-dimensional k-vector space V such that G' is the scheme-theoretic stabilizer of a line. The proof of this result in [Bo2, 5.1] works without smoothness by using points valued in artin local rings (not just fields); see [CGP, Prop. A.2.4]. ■

The analogue of Lemma 4.2.5 using $H(\mathbf{A}_k)$ and $H'(\mathbf{A}_k)$ instead of $H(\mathbf{A}_k)^1$ and $H'(\mathbf{A}_k)^1$ is false. For example, let G be a nontrivial k-split connected semisimple k-group and P a proper parabolic k-subgroup, and consider $H = P$ and $H' = G$. The natural continuous open map $H'(\mathbf{A}_k)/H(\mathbf{A}_k) \to (G/P)(\mathbf{A}_k)$ is a homeomorphism (because of the standard fact that $G(F)/P(F) = (G/P)(F)$ for any field F/k, such as $F = k_v$). But $(G/P)(\mathbf{A}_k)$ is compact since G/P is projective, so $H'(k)H(\mathbf{A}_k)$ is not closed in $H'(\mathbf{A}_k)$ since otherwise the subset $G(k)/P(k) \subseteq (G/P)(\mathbf{A}_k)$ would admit a structure of compact Hausdorff space, an impossibility since it is countably infinite (as the countable $G(k)$ is Zariski-dense in G, and $P \neq G$).

The following standard notion allows us to extend the concept of a purely inseparable isogeny between smooth groups of finite type over a field to cases in which smoothness does not hold.

Definition 4.2.6. A map of schemes $f : Y \to Z$ is *radiciel* if it is injective and induces a purely inseparable extension on residue fields $\kappa(f(y)) \to \kappa(y)$ for all $y \in Y$.

A surjective map between finite type schemes over a field F is radiciel precisely when it induces a bijection on $\overline{F}$-points (with $\overline{F}$ an algebraic closure of F), and for a surjective *finite* map between connected normal F-schemes of finite type it is equivalent to say that the extension of function fields is purely inseparable.

Lemma 4.2.7. *For any affine k-group scheme G of finite type and any finite non-empty set S of places of k containing the archimedean places, the subgroup $G(\mathbf{A}_k)^1 \cdot G(k_S)$ in $G(\mathbf{A}_k)$ has finite index.*

Proof. We initially give an argument that works in characteristic 0, and then modify it for nonzero characteristic (using the discreteness of the idelic norm) to circumvent difficulties caused by radiciel k-homomorphisms. Assume first that G is smooth and connected, with no hypotheses on $\mathrm{char}(k)$, so $\mathrm{X}_k(G)$ is a finite free $\mathbf{Z}$-module. Let T be the split k-torus $\mathrm{X}_k(G)^\vee \otimes_{\mathbf{Z}} \mathrm{GL}_1$ (i.e., the k-torus with character group $\mathrm{X}_k(G)$). The natural map $G \to T$ is the unique maximal k-split torus quotient, and this map identifies $\mathrm{X}_k(T)$ with $\mathrm{X}_k(G)$ in the natural manner. We thereby obtain a natural injection of abelian groups

$$G(\mathbf{A}_k)/G(\mathbf{A}_k)^1 \to T(\mathbf{A}_k)/T(\mathbf{A}_k)^1.$$

Consider the commutative diagram of groups

$$\begin{array}{ccc} G(\mathbf{A}_k)/G(\mathbf{A}_k)^1 & \longrightarrow & T(\mathbf{A}_k)/T(\mathbf{A}_k)^1 \\ \uparrow & & \uparrow \\ G(k_S) & \longrightarrow & T(k_S) \end{array}$$

We need to prove that the map along the left has image with finite index, so by injectivity of the top row it is enough to prove the maps along the bottom and right sides have images with finite index.

First we check that the cokernel along the right side is finite. Since T is split, we only have to consider the analogue for GL_1. This case is obvious by separately considering number fields and function fields (using that S contains archimedean places in the number field case and is not empty in the function field case).

By Lemma 4.1.2 applied to the smooth k_v-group G_{k_v} for each $v \in S$, the map $G(k_S) \to T(k_S)$ has image with finite index as long as the scheme-theoretic kernel of the quotient map $G \to T$ is smooth. This is automatic in characteristic 0, so the case of number fields is settled for connected G. To settle the general number field case (for which G is automatically smooth), we give an argument to reduce the general case to the connected case without smoothness hypotheses on G or any hypotheses on $\mathrm{char}(k)$.

As we noted at the beginning of §3.2, there is a finite set S' of places such that $S \subseteq S'$ and G spreads out to an affine finite type $\mathscr{O}_{k,S'}$-group $G_{S'}$ containing an open and closed normal subgroup $G^0_{S'}$ with generic fiber G^0. For any finite S'' containing S', let $\mathbf{A}_{k,S''} \subset \mathbf{A}_k$ denote the open subring $(\prod_{v\in S''} k_v) \times \prod_{v\notin S''} \mathscr{O}_v$. (Don't confuse this with the factor ring $\mathbf{A}_k^{S''}$.) The compact space $G(\mathbf{A}_k)/G^0(\mathbf{A}_k)$ (see Proposition 3.2.1) is the rising union of open subsets $G_{S'}(\mathbf{A}_{k,S''})/G^0_{S'}(\mathbf{A}_{k,S''})$ for finite S'' containing S', so exhaustion is attained for large enough S'' that we may and do rename as S'. Since $G^0(k_S)$ has finite index in $G(k_S)$, if $G^0(k_S)$ has finite-index image in $G^0(\mathbf{A}_k)/G^0(\mathbf{A}_k)^1$ then we just have to show that

$$G_{S'}(\mathbf{A}_{k,S'})/G^0_{S'}(\mathbf{A}_{k,S'})G(k_S)G_{S'}(\mathbf{A}_{k,S'})^1 \tag{4.2.1}$$

is finite, where $G_{S'}(\mathbf{A}_{k,S'})^1 := G_{S'}(\mathbf{A}_{k,S'}) \cap G(\mathbf{A}_k)^1$. The compact factor $\prod_{v\notin S'} G_{S'}(\mathscr{O}_v)$ is killed in (4.2.1), so (4.2.1) is a quotient of $G(k_{S'})/G^0(k_{S'}) \subseteq (G/G^0)(k_{S'})$, which is finite. Hence, we may replace G with G^0.

It remains to consider the general case when k has characteristic $p > 0$. By the preceding general argument resting on Proposition 3.2.1 we may and do now assume that G is connected. If G is smooth then we have to address the possibility that the map $G \twoheadrightarrow T$ onto the maximal k-split torus quotient may have non-smooth kernel. Let q be the size of the constant field in k, so the idelic norm on $\mathbf{A}_k^\times$ has image $q^{\mathbf{Z}}$. The group $G(\mathbf{A}_k)/G(\mathbf{A}_k)^1$ is a subgroup of the finite free $\mathbf{Z}$-module $\mathrm{Hom}(\mathrm{X}_k(G), q^{\mathbf{Z}})$, so it is also a finite free $\mathbf{Z}$-module and hence the abelian group $G(\mathbf{A}_k)/G(\mathbf{A}_k)^1 G(k_S)$ is finite if it is killed by some nonzero integer. Thus, for smooth G, instead proving that $G(k_v) \to T(k_v)$ has image with finite index for each $v \in S$ it suffices to prove that the cokernel is killed by some nonzero integer.

When G is smooth, for a maximal k-torus $T' \subseteq G$ the map $T' \to T$ is a surjection of k-tori [Bo2, 11.14]. (If G is not smooth then such a T' surjecting onto T may not exist.) Surjections between tori over a field are split in the isogeny category over the field, so the cokernel of $T'(k_v) \to T(k_v)$ is killed by a nonzero integer for all v. Hence, the case of smooth *connected* G over k is settled, so the smooth case over k is settled.

The general case over function fields is reduced to the smooth case as follows. Let $G' \subseteq G$ be as in Lemma 3.1.1. (This may be disconnected even if G is connected; see [CGP, Rem. C.4.2].) Since $G'(\mathbf{A}_k) \to G(\mathbf{A}_k)$ is an isomorphism that carries $G'(\mathbf{A}_k)^1$ into $G(\mathbf{A}_k)^1$ by functoriality, $G(\mathbf{A}_k)/G(\mathbf{A}_k)^1$ is a quotient of $G'(\mathbf{A}_k)/G'(\mathbf{A}_k)^1$. The equality $G'(k_S) = G(k_S)$ therefore reduces us to showing that $G'(k_S)$ has finite-index image in $G'(\mathbf{A}_k)/G'(\mathbf{A}_k)^1$, and this holds since G' is smooth. ■

5. Proof of finiteness of class numbers (Theorem 1.3.1)

By §3, to prove the finiteness of class numbers for all affine group schemes of finite type over a global function field k it is enough to restrict attention to smooth affine k-groups G, and passing to G^0 is harmless (Corollary 3.2.2). We now assume that G is a smooth connected affine k-group and will use a quotient presentation of G over k to reduce the finiteness of class numbers for G

to the case of pseudo-reductive groups. We first review the known connected reductive case, to clarify the ideas.

5.1 Finiteness in the reductive case

In this section, assume that G is connected and reductive over a global function field k. Let Z be its maximal central torus and $\mathscr{D}(G)$ its semisimple derived group. The strategy in the connected reductive case is to reduce the problem to the case of simply connected groups. The special arithmetic features of simply connected groups over global function fields are triviality of degree-1 Galois cohomology and strong approximation.

Finiteness of class numbers for commutative G is Example 1.3.2, so we may assume $\mathscr{D}(G) \neq 1$. Let $\{G_i\}$ be the non-empty finite set of minimal nontrivial smooth connected normal k-subgroups of $\mathscr{D}(G)$; they are k-simple and pairwise commute. For the simply connected central cover $\widetilde{G}_i \to G_i$, the multiplication map

$$H := Z \times \prod_i \widetilde{G}_i \to G$$

is a k-isogeny with finite central multiplicative kernel μ. The degree-1 local and global cohomology of μ over k (for the *fppf* topology) may be infinite, so we need to pass to another quotient presentation for G.

Let T be a maximal k-torus in $Z \times \prod_i \widetilde{G}_i$. (We have $\mu \subseteq T$ since a maximal torus in a connected reductive group is its own scheme-theoretic centralizer.) The T-action on H via $t.h = tht^{-1}$ factors through an action by the central quotient T/μ, and this action by T/μ on H is trivial on the k-subgroup $T \subseteq H$. The resulting twisted diagonal homomorphism $h : T \to E := H \rtimes (T/\mu)$ analogous to (2.1.2) is a closed k-subgroup inclusion that makes T a central torus in E. The natural homomorphism $H \to \mathrm{coker}(h)$ between smooth k-groups is surjective with scheme-theoretic kernel μ, so $G \simeq H/\mu \simeq \mathrm{coker}(h)$. Thus, E is a central extension of our initial connected reductive group G by the k-torus T, and E is also a semidirect product of the k-torus $\mathscr{T} = Z \times (T/\mu)$ against a product $P := \prod \widetilde{G}_i$ of simply connected and k-simple connected semisimple k-groups $\widetilde{G}_i$. This is summarized by the diagram of exact sequences

(5.1.1)
$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ & & & & P & & & & \\ & & & & \downarrow & & & & \\ 1 & \longrightarrow & T & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & & & \downarrow & & & & \\ & & & & \mathscr{T} & & & & \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \end{array}$$

in which the vertical sequence splits as a semi-direct product. The derived group $\mathscr{D}(E)$ is the simply connected k-subgroup $P = \prod \widetilde{G}_i \subseteq E$ since $\mathscr{T}$ is commutative.

The following important result is largely due to Kneser-Bruhat-Tits and Harder:

Theorem 5.1.1. *Let k be a field and G a smooth connected affine k-group. Assume that G is either reductive or basic exotic pseudo-reductive* (*with* $\mathrm{char}(k) \in \{2, 3\}$).

(i) *Assume k is a global function field or a non-archimedean local field. If G either is semisimple and simply connected or is basic exotic then* $\mathrm{H}^1(k, G) = 1$.

(ii) *Assume k is a global field. The k-group G is k-anisotropic if and only if $G(k)\backslash G(\mathbf{A}_k)$ is compact.*

Proof. By Theorem 2.3.8(ii), the basic exotic case reduces to the simply connected semisimple case. It remains to consider reductive G. In this case, (i) is due to Kneser–Bruhat–Tits [BTIII, Thm. 4.7(ii)] for non-archimedean local k and Harder [Ha2, Satz A] for global function fields. Likewise, (ii) is due to Harder [Ha1, 2.2.7(i),(ii)] over global function fields in the reductive case. Over number fields, (ii) is an immediate consequence of the general fact (for connected reductive G) that $G(k)\backslash G(\mathbf{A}_k)^1$ is compact if and only if all k-split tori in G are central; see the discussion early in Appendix A.5 (including Proposition A.5.1). ■

Let S be a finite non-empty set of places of a global function field k. We now review why $P = \prod_i \widetilde{G}_i$ has finite class numbers. Since each $\widetilde{G}_i$ is the Weil restriction of an absolutely simple and simply connected group over a finite separable extension of k [BoT1, 6.21(ii)] (or see [CGP, Prop. A.5.14]), we easily reduce to the case when P is absolutely simple and simply connected. If the simply connected P is anisotropic over k then $P(k)\backslash P(\mathbf{A}_k)$ is compact, by Theorem 5.1.1(ii), so finiteness of class numbers for P is obvious in such cases. If P is k-isotropic then $P(k_v)$ is non-compact for all $v \in S$, so by the strong approximation theorem for (connected and) absolutely simple and simply connected groups over global fields [Pr1, Thm. A] the subgroup $P(k) \subseteq P(\mathbf{A}_k^S)$ is dense. Hence, again finiteness of class numbers for P is clear.

The degree-1 Galois cohomology of T may be nontrivial (over k and its completions), so to avoid the serious difficulties that this can cause we have to replace T with a better torus, as follows. Let k'/k be a finite separable extension that splits T, so there is a closed immersion of k-tori $T \hookrightarrow T' = \mathrm{R}_{k'/k}(T_{k'})$, and T' has vanishing degree-1 cohomology over k and every completion k_v. Since T is central in E, we can form the pushout $E' = (E \times T')/T$ (with T embedded by the twisted diagonal map $t \mapsto (t^{-1}, t)$). This pushout contains P as a normal closed k-subgroup (since the scheme-theoretic intersection of $P \subseteq E$ and T inside of $E \times T'$ is trivial), so there is a pair of exact sequences of k-groups

$$1 \to P \to E' \to T'' \to 1 \tag{5.1.2}$$

(with $T'' = (\mathscr{T} \times T')/T$ commutative, even a torus) and

$$1 \to T' \to E' \to E/T = G \to 1. \tag{5.1.3}$$

These sequences recover (1.4.1) for G since $\mathscr{D}(E') = P$ and the k-group P is simply connected. The latter property of P will help us to handle the fact that (5.1.2) does not generally split as a semi-direct product.

We shall use (5.1.2) to prove that E' has finite class numbers and then feed this into (5.1.3) to deduce via the cohomological properties of T' that the connected reductive G has finite class numbers By using Theorem 5.1.1(i), standard "spreading out" arguments, and Lang's theorem (cf. proof of Lemma A.2.1), the exact sequence (5.1.2) induces an exact sequence of topological groups

$$1 \to P(\mathbf{A}_k^S) \to E'(\mathbf{A}_k^S) \to T''(\mathbf{A}_k^S) \to 1$$

with open projection map to the quotient as well as an exact sequence of abstract groups

$$1 \to P(k) \to E'(k) \to T''(k) \to 1.$$

In particular, if we pick a compact open subgroup $K \subseteq E'(\mathbf{A}_k^S)$ then its image K'' in $T''(\mathbf{A}_k^S)$ is a compact open subgroup, so by finiteness of class numbers for T'' there is a finite set of elements $t_i \in T''(\mathbf{A}_k^S)$ such that $T''(\mathbf{A}_k^S) = \cup K'' t_i T''(k)$. Pick $e_i' \in E'(\mathbf{A}_k^S)$ mapping to t_i. Since $E'(k)$ surjects onto $T''(k)$ we get

$$E'(\mathbf{A}_k^S) = \bigcup_i K e_i' E'(k) P(\mathbf{A}_k^S) = \bigcup_i K e_i' P(\mathbf{A}_k^S) E'(k).$$

By finiteness of class numbers for P, for the compact open subgroup $\widetilde{K}_i = (e_i')^{-1}(K \cap P(\mathbf{A}_k^S))e_i'$ in $P(\mathbf{A}_k^S)$ there is a finite set of elements $h_{ij} \in P(\mathbf{A}_k^S)$ $(j \in J_i)$ such that $P(\mathbf{A}_k^S) = \cup_j \widetilde{K}_i h_{ij} P(k)$, so

$$E'(\mathbf{A}_k^S) = \bigcup_{i,j} K e_i' \widetilde{K}_i h_{ij} E'(k) = \bigcup_{i,j} K e_i' h_{ij} E'(k).$$

This gives quasi-compactness of $E'(k)\backslash E'(\mathbf{A}_k^S)$, so E' has finite class numbers.

The k'-torus T' has trivial degree-1 cohomology over the completions k_v, so $E'(\mathbf{A}_k^S) \to G(\mathbf{A}_k^S)$ via (5.1.3) is surjective. Thus, $G(k)\backslash G(\mathbf{A}_k^S)$ is a continuous image of $E'(k)\backslash E'(\mathbf{A}_k^S)$, so it is quasi-compact. Thus, finiteness of class numbers for connected reductive groups over global function fields is proved.

We now adapt the preceding argument so that it works under some axioms that will be applicable in our general proof of finiteness of class numbers in the function field case.

Theorem 5.1.2. *Let G be a smooth connected affine group over a global field k. Let N be a solvable smooth connected normal k-subgroup of G such that $\overline{G} := G/N$ has finite class numbers. If the open image of $G(k_v) \to \overline{G}(k_v)$ has finite index for all places v then G has finite class numbers.*

The finite-index hypothesis holds if $\mathrm{H}^1(k_v, N)$ is finite for all v, but this cohomology can be infinite in the function field case, even for commutative pseudo-reductive N; see [CGP, Ex. 11.3.3]. The toric criterion in Proposition 4.1.9 will be especially useful to verify the finite-index hypothesis.

Proof. Let S be a finite non-empty set of places of k containing the archimedean places. Choose a compact open subgroup K in $G(\mathbf{A}_k^S)$ and let $\overline{K}$ be its compact image in $\overline{G}(\mathbf{A}_k^S)$. The map $G(\mathbf{A}_k^S) \to \overline{G}(\mathbf{A}_k^S)$ is open since N is smooth and *connected*, so the compact subgroup $\overline{K}$ is open. Since $\overline{G}$ has finite class numbers by hypothesis, there exists a finite set of elements $y_i \in \overline{G}(\mathbf{A}_k^S)$ such that

$$\overline{G}(\mathbf{A}_k) = \coprod_i \overline{G}(k) y_i \overline{G}(k_S) \overline{K}; \tag{5.1.4}$$

note that $\overline{K}$ and $\overline{G}(k_S)$ commute since $\mathbf{A}_k = k_S \times \mathbf{A}_k^S$.

The k-subgroup N in G satisfies the requirements in Theorem A.1.1, so the natural map

$$\mathrm{pr}^0 : G(k)\backslash G(\mathbf{A}_k)^1 \to \overline{G}(k)\backslash \overline{G}(\mathbf{A}_k)$$

is proper. Hence, for any $y \in \overline{G}(k_S)$ the preimage

$$(\mathrm{pr}^0)^{-1}(\overline{G}(k)\backslash \overline{G}(k) y_i y \overline{K}) \subseteq G(k)\backslash G(\mathbf{A}_k)^1 \tag{5.1.5}$$

is compact in $G(k)\backslash G(\mathbf{A}_k)^1$.

Since K is a compact open subgroup in $G(\mathbf{A}_k^S)$, the product set $G(k_S) \cdot K = K \cdot G(k_S)$ is an open subgroup of $G(\mathbf{A}_k)$. Choose a set $\{\overline{g}_\ell\}$ of coset representatives in $\overline{G}(k_S)$ modulo right multiplication by the image of $G(k_S)$, so for each y_i in (5.1.4) the compact preimage in (5.1.5) for $y = \overline{g}_\ell$ is contained in a union of finitely subsets $G(k)\backslash G(k)g_{ij\ell}G(k_S)K \subseteq G(k)\backslash G(\mathbf{A}_k)$. There are only finitely many $\overline{g}_\ell$'s because the open image of $G(k_v)$ in $\overline{G}(k_v)$ has finite index for all $v \in S$ by hypothesis. We have $K \subseteq G(\mathbf{A}_k)^1$ since K is compact, so $G(\mathbf{A}_k)^1 G(k_S)$ is the union of the finitely many double cosets

$$G(k)g_{ij\ell}G(k_S)K.$$

(Keep in mind that $G(k_S)$ commutes with K and that $\overline{G}(k_S)$ commutes with $\overline{K}$.)

By Lemma 4.2.7, $G(\mathbf{A}_k)^1 G(k_S)\backslash G(\mathbf{A}_k)$ admits a finite set $\{x_r\}$ of representatives in $G(\mathbf{A}_k^S)$. Let $\widetilde{K} \subseteq G(\mathbf{A}_k^S)$ denote the compact open subgroup $\bigcap_r x_r K x_r^{-1}$. If we go through the preceding argument again but use $\widetilde{K}$ in the role of K (so the set of $g_{ij\ell}$'s will change: now we express $G(\mathbf{A}_k)^1 G(k_S)$ as a union of finitely many double cosets $G(k)g_{ij\ell}G(k_S)\widetilde{K}$), then for any $g \in G(\mathbf{A}_k)$ we can write $g = g_r x_r$ for a unique r and a unique $g_r \in G(\mathbf{A}_k)^1 G(k_S)$. Since $g_r \in G(k)g_{ij\ell}G(k_S)\widetilde{K}$ for some $g_{ij\ell}$, we have

$$g \in G(k)g_{ij\ell}G(k_S)\widetilde{K}x_r \subseteq G(k)g_{ij\ell}G(k_S)x_r K = G(k)g_{ij\ell}x_r G(k_S)K$$

because the element $x_r \in G(\mathbf{A}_k^S)$ commutes with $G(k_S)$. Thus, the finite set of products $g_{ij\ell}x_r$ represents all elements of the double coset space $\Sigma_{G,S,K}$. ■

5.2 Finiteness in the pseudo-reductive case

Now we prove finiteness of class numbers when G is a pseudo-reductive k-group and $\mathrm{char}(k) > 0$. By Theorem 2.3.6(ii) and Theorem 2.3.8, it suffices to treat the case when G is a generalized standard pseudo-reductive k-group.

If G is commutative then finiteness of class numbers for G was established in Example 1.3.2. Now assume that G is a non-commutative generalized standard pseudo-reductive k-group, and let T be a maximal k-torus in G and $C = Z_G(T)$ the corresponding Cartan k-subgroup. By Remark 2.3.4 there is a generalized standard presentation $(G', k'/k, T', C)$ of G adapted to T, providing an isomorphism

$$G \simeq (\mathrm{R}_{k'/k}(G') \rtimes C)/\,\mathrm{R}_{k'/k}(C') \tag{5.2.1}$$

where $C' = Z_{G'}(T')$ is a commutative Cartan k'-subgroup of G'. The fiber G'_i of G' over each factor field k'_i of k' is either a connected semisimple k'_i-group that is absolutely simple and simply connected or is a basic exotic k'_i-group. Hence, each G'_i has finite class numbers: in the simply connected semisimple case this follows from strong approximation and adelic compactness results, as we reviewed in §5.1, and in the basic exotic case it is reduced to the simply connected semisimple case by Theorem 2.3.8(ii).

The finiteness of class numbers for $\mathrm{R}_{k'/k}(G')$ follows from such finiteness for the fibers of G' over the factor fields of k'. Let Z denote the central subgroup $\mathrm{R}_{k'/k}(C')$ in $H := \mathrm{R}_{k'/k}(G') \rtimes C$. The technique used for connected reductive groups over function fields in §5.1 will now be adapted to prove finiteness of class numbers for the pseudo-reductive group $G = H/Z$.

There is a finite extension F/k such that the smooth connected commutative F-group Z_F has an F-split maximal torus and an F-split unipotent quotient modulo this torus. Thus, Z_F has trivial degree-1 cohomology over F and its completions. The Weil restriction $\mathscr{Z} = \mathrm{R}_{F/k}(Z_F)$ is a smooth connected commutative k-group with the analogous cohomological properties over k and its completions (by Lemma 4.1.6), and the natural map $Z \to \mathscr{Z}$ is a closed subgroup inclusion that gives rise to the central pushout $\mathscr{E} = (H \times \mathscr{Z})/Z$. Consider the pair of exact sequences

$$1 \to \mathrm{R}_{k'/k}(G') \to \mathscr{E} \to (C \times \mathscr{Z})/Z \to 1 \tag{5.2.2}$$

and

$$1 \to \mathscr{Z} \to \mathscr{E} \to H/Z = G \to 1 \tag{5.2.3}$$

that are respectively analogous to (5.1.2) and (5.1.3). In particular, by Theorem 5.1.1(i) the local and global degree-1 Galois cohomologies for $\mathrm{R}_{k'/k}(G')$ are trivial (due to the fact that the factor fields of k' are global function fields). The quotient $(C \times \mathscr{Z})/Z$ has finite class numbers (as it is commutative). Thus, we can use the method of analysis of (5.1.2) and (5.1.3) to first deduce finiteness of class numbers for $\mathscr{E}$ from such finiteness for $\mathrm{R}_{k'/k}(G')$ and $(C \times \mathscr{Z})/Z$, and then use this finiteness property for $\mathscr{E}$ to deduce the same for its quotient $\mathscr{E}/\mathscr{Z} \simeq H/Z = G$ via the cohomological properties of $\mathscr{Z}$.

5.3 Another application of pseudo-reductive structure theory

Finally, we treat general (smooth connected) G. Let $U \subseteq G$ be the maximal smooth connected unipotent normal k-subgroup, so $Q := G/U$ is pseudo-reductive over k. By §5.2, Q has finite class numbers. Thus, Theorem 5.1.2 (which rests on Theorem A.1.1, whose proof over function fields is much harder than over number fields) can be applied provided that the open map $G(k_v) \to Q(k_v)$ has image with finite index for all places v of k.

To establish that the open image $\mathscr{U}_v$ of $G(k_v)$ in $Q(k_v)$ has finite index for each v, let $\overline{T}_v$ be a maximal k_v-split torus in Q_{k_v}. Since Q is pseudo-reductive over k, so Q_{k_v} is pseudo-reductive over k_v (as k_v/k is separable), by Proposition 4.1.9 it suffices to show that $\mathscr{U}_v$ meets $\overline{T}_v(k_v)$ with finite index. By Proposition 3.1.3, there exists a maximal k_v-split torus T'_v in G_{k_v} mapping onto $\overline{T}_v$. The induced map $T'_v(k_v)/T'_v(k_v)^1 \to \overline{T}_v(k_v)/\overline{T}_v(k_v)^1$ on quotients modulo the maximal compact subgroups has image with finite index (Lemma 4.1.2). The open subgroup $\mathscr{U}_v \subseteq Q(k_v)$ must meet the compact subgroup $\overline{T}_v(k_v)^1$ in a finite-index subgroup of $\overline{T}_v(k_v)^1$, so $\mathscr{U}_v \cap \overline{T}_v(k_v)$ has finite index in $\overline{T}_v(k_v)$.

6. Proof of finiteness of Ш (Theorem 1.3.3)

Our proof of Theorem 1.3.3 will be characteristic-free, up to replacing the condition "$S \neq \emptyset$" with the condition "$S \supseteq S_\infty$".

We begin by reviewing a standard argument to deduce Theorem 1.3.3(ii) from Theorem 1.3.3(i). Let X be a k-scheme equipped with a right action by an affine k-group scheme G of finite type. Fix a point $x \in X(k)$ and let $G_x \subseteq G$ be the stabilizer subgroup scheme of x over k. That is, G_x is the pullback of the diagonal $\Delta_{X/k} : X \to X \times X$ under the map $G \to X \times X$ defined by $g \mapsto (x.g, x)$. Consider $x' \in X(k)$ such that x' is $G(k_v)$-conjugate to x in $X(k_v)$ for all $v \notin S$, and let $H_{x',x}$ be the subscheme of G consisting of points carrying x' to x. (That is, for any k-algebra R, $H_{x',x}(R)$ is the set of $g \in G(R)$ such that $x'.g = x$ in $X(R)$, so $H_{x',x}$ is the pullback of the diagonal $\Delta_{X/k}$ under the map $G \to X \times X$ defined by $g \mapsto (x'.g, x)$.) There is an evident right action of G_x on $H_{x',x}$ over k, and for any place $v \notin S$ we see that the subscheme $(H_{x',x})_{k_v} \subseteq G_{k_v}$ is a left $G(k_v)$-translate of $(G_x)_{k_v}$. In particular, $H_{x',x}$ is a right G_x-torsor over k (for the *fppf* topology over k), and as such it is trivial over k_v for all $v \notin S$. Since each k_v/k is separable, these torsors are even locally trivial for the étale topology over k.

If x'' is a second such point and there is a k-isomorphism $H_{x'',x} \simeq H_{x',x}$ as abstract G_x-

torsors over k (contained in G) then by descent theory such a k-isomorphism must be given by left multiplication on G by some $g \in G(k)$. Thus, $H_{x'.g,x} = H_{x'',x}$ inside of G. For any $\overline{k}$-point h of this common subscheme, $x'' = x''.(hh^{-1}) = x.h^{-1} = x'.g$ in $X(\overline{k})$. That is, $x'.g = x''$ inside of $X(k)$. Hence, it suffices to prove that there are only finitely many k-isomorphism classes of right G_x-torsors that are trivial over k_v for all $v \notin S$. (It is equivalent to consider such torsors for the *fppf* or étale topologies over k.) That is, we just need finiteness of $Ш^1_S(k, G_x)$. This is a special case of the finiteness result in part (i) of Theorem 1.3.3, so we may now focus our efforts on proving part (i).

6.1 Reduction to the smooth case

We turn to the task of proving Theorem 1.3.3(i), so k is a global function field and G is an affine k-group scheme of finite type. The method of cohomological twisting will be used, so let us review this technique. For $c \in \mathrm{H}^1(k, G)$ represented by a right G-torsor Y over k, we have the associated inner form ${}_YG = \underline{\mathrm{Aut}}_G(Y)$ of G as in Appendix B.1. There is a commutative diagram of sets

$$\begin{array}{ccc} \mathrm{H}^1(k,G) & \xrightarrow{\theta_{S,G}} & \prod_v \mathrm{H}^1(k_v,G) \\ {\scriptstyle t_{Y,k}}\Big\downarrow\simeq & & \simeq\Big\downarrow{\scriptstyle \prod t_{Y,k_v}} \\ \mathrm{H}^1(k,{}_YG) & \xrightarrow[\theta_{S,{}_YG}]{} & \prod_v \mathrm{H}^1(k_v,{}_YG) \end{array}$$

in which the vertical twisting maps are as defined in Appendix B.2, where it is also proved that these twisting maps are bijective. Thus, the set $\theta_{S,G}^{-1}(\theta_{S,G}(c))$ is in bijection with $\ker \theta_{S,{}_YG}$, so to prove the finiteness of fibers of $\theta_{S,G}$ in general it suffices (after renaming ${}_YG$ as G) to prove finiteness of the fiber $Ш^1_S(k, G) := \ker \theta_{S,G}$ over the distinguished point in general.

By Lemma 3.1.1, there is a unique smooth closed k-subgroup $G' \subseteq G$ such that $G'(K) = G(K)$ for every separable extension field K/k, so the following lemma reduces our problem to the case of smooth groups.

Lemma 6.1.1. *The natural map* $\mathrm{H}^1(k, G') \to \mathrm{H}^1(k, G)$ *carries* $Ш^1_S(k, G')$ *isomorphically onto* $Ш^1_S(k, G)$.

The map $\mathrm{H}^1(k, G') \to \mathrm{H}^1(k, G)$ is generally not surjective (e.g., consider an infinitesimal group scheme G, such as μ_p in characteristic $p > 0$). Also keep in mind that G' may be disconnected even if G is connected.

Proof. This is proved in [CGP, Ex. C.4.3]. The idea (as in the proof of [GM, Prop. 3.1]) is to show that an inverse is given at the level of torsors by assigning to any right G-torsor E over k the right G'-torsor E' as in Lemma 3.1.1; the local triviality of E is needed to prove that E' really is a G'-torsor (e.g., $E' \neq \emptyset$). ∎

6.2 Reduction to the connected case

Since we have reduced our finiteness problem to the case of *smooth* affine k-groups G, we may identify the set $\mathrm{H}^1(k, G)$ of isomorphism classes of right G-torsors over k with the degree-1 Galois cohomology set $\mathrm{H}^1(k_s/k, G)$ for a fixed choice of separable closure k_s of the global field k. We also fix separable closures $k_{v,s}$ and embeddings $k_s \to k_{v,s}$ over $k \to k_v$ for all places v of k when we need to work with restriction maps to local Galois cohomology.

Assume that the finiteness of $Ш^1_S(k,H)$ is known for all smooth *connected* affine k-groups H and all choices of S, and let us prove it in general for any smooth affine k-group G and any choice of S. The method we will use is a variant of the argument of Borel and Serre in [BS, §7]. Since they appeal to characteristic 0 (via finiteness of $\mathrm{H}^1(k_v,G)$), we prefer to give an argument that works in any characteristic.

Let $G^0 \subseteq G$ be the identity component and let $\Gamma = G/G^0$ be the finite étale component group of G over k, so we have an exact sequence of smooth k-groups

$$1 \to G^0 \xrightarrow{j} G \xrightarrow{\pi} \Gamma \to 1 \tag{6.2.1}$$

with G^0 connected and Γ finite. We can increase S since $Ш^1_S(k,G) \subseteq Ш^1_{S'}(k,G)$ when S' contains S. By standard "spreading out" arguments that are explained in Appendix A.2, if we increase S then we can arrange that S is non-empty and that the exact sequence (6.2.1) is the generic fiber of a short exact sequence

$$1 \to G^0_S \to G_S \to \Gamma_S \to 1$$

of smooth affine $\mathscr{O}_{k,S}$-groups with finite étale Γ_S and an $\mathscr{O}_{k,S}$-group G^0_S whose fibers are *connected*. Thus, the induced sequence on $\mathscr{O}_v$-points is exact for all $v \notin S$, by Lang's theorem. But for such v we have

$$\Gamma(k_v) = \Gamma_S(k_v) = \Gamma_S(\mathscr{O}_v)$$

since Γ_S is $\mathscr{O}_{k,S}$-finite, so $\pi : G(\mathbf{A}^S_k) \to \Gamma(\mathbf{A}^S_k)$ is surjective.

By an application of the Chebotarev density theorem, $Ш^1_S(k,\Gamma)$ is finite since Γ is a finite étale k-group [BS, Lemme 7.3], so the natural map $f : Ш^1_S(k,G) \to Ш^1_S(k,\Gamma)$ has finite target. Thus, the finiteness of $Ш^1_S(k,G)$ is equivalent to finiteness of the non-empty fibers of the map f. That is, we choose $c \in Ш^1_S(k,G)$ and wish to prove finiteness of $f^{-1}(f(c))$. By choosing a Galois cocycle in $\mathrm{Z}^1(k_s/k,G) \subseteq G(k_s \otimes_k k_s)$ that represents c (or more conceptually, choosing a right G-torsor that represents c) we get an associated inner form of G. Since G naturally acts on both G^0 and Γ, we can adapt this inner form construction as in [Se2, I, §5.3] to compatibly twist both the normal subgroup G^0 and the quotient Γ. The abstract k-isomorphism classes of these resulting k-forms of G, G^0, and Γ only depend on c, but for functoriality purposes we must use a common choice of cocycle representative for c when performing the twisting constructions. Nonetheless, we abuse notation by writing G_c, $(G^0)_c$, and Γ_c to denote these k-forms.

The k-form $(G^0)_c$ of G^0 is identified with the identity component of G_c, and there is a "c-twisted" k-homomorphism $\pi_c : G_c \to \Gamma_c$ that is identified with the projection onto the étale component group of G_c. Beware that G^0_c is generally not an inner form of G^0, so although we have natural bijections between the global (resp. local) degree-1 cohomologies of G and its inner form G_c [Se2, I, §5.3, Prop. 35bis] we do not have the same for G^0 and G^0_c; cf. [Se2, I, §5.5, Rem.]. However, Γ_c is an inner form of Γ, so we do have such bijections for Γ and Γ_c and thus we have a commutative diagram of sets

$$\begin{array}{ccc} Ш^1_S(k,G) & \xrightarrow{f} & Ш^1_S(k,\Gamma) \\ \simeq\downarrow & & \downarrow\simeq \\ Ш^1_S(k,G_c) & \xrightarrow[f_c]{} & Ш^1_S(k,\Gamma_c) \end{array}$$

in which the vertical maps are bijective and the left side carries c to the trivial point in $Ш^1_S(k,G_c)$. Hence, by replacing G with G_c (and f with f_c) it suffices to prove the finiteness of $\ker f$.

Define the subset

$$\coprod_{v \notin S} \mathrm{H}^1(k_v, G) \subseteq \prod_{v \notin S} \mathrm{H}^1(k_v, G) \tag{6.2.2}$$

to consist of tuples (c_v) such that the element $c_v \in \mathrm{H}^1(k_v, G)$ is the distinguished point for all but finitely many v; if G is commutative then this is the direct sum inside of a direct product.

Lemma 6.2.1. *The localization map θ_{S,G^0} has image contained inside of $\coprod_{v \notin S} \mathrm{H}^1(k_v, G^0)$.*

Proof. This is [Oes, IV, 2.4, Cor.] for affine G^0; the proof there even works for arbitrary smooth connected k-groups. Alternatively, a short direct proof in terms of torsors goes as follows. By Remark 1.2.1, the elements of $\mathrm{H}^1(k, G^0)$ classify isomorphism classes of k-scheme G^0-torsors X. For any such X there is a finite non-empty set S' of places of k (containing S) such that G^0 spreads out to a smooth $\mathscr{O}_{k,S'}$-group $\mathscr{G}$ of finite type with connected fibers and X spreads out to a $\mathscr{G}$-torsor $\mathscr{X}$ over $\mathscr{O}_{k,S'}$. By Lang's theorem, for all $v \notin S'$ the fiber of $\mathscr{X}$ over the residue field at v must have a rational point. This lifts to $\mathscr{X}(\mathscr{O}_v)$ by smoothness, so $X(k_v) \neq \emptyset$ for all $v \notin S'$. ■

An element $c \in Ш^1_S(k, G) \subseteq \mathrm{H}^1(k, G)$ lies in $\ker f$ if and only if it is in the image of the natural map $j : \mathrm{H}^1(k, G^0) \to \mathrm{H}^1(k, G)$. Thus, for an element $c^0 \in \mathrm{H}^1(k, G^0)$ we have $j(c^0) \in \ker f$ if and only if the element $\theta_{S,G^0}(c^0) \in \coprod_{v \notin S} \mathrm{H}^1(k_v, G^0)$ maps to the distinguished element in $\coprod_{v \notin S} \mathrm{H}^1(k_v, G)$.

For each place $v \notin S$ we have an exact sequence of pointed sets

$$G(k_v) \xrightarrow{\pi_v} \Gamma(k_v) \xrightarrow{\delta_v} \mathrm{H}^1(k_v, G^0) \xrightarrow{j_v} \mathrm{H}^1(k_v, G),$$

so passing to the "direct sum" and using that $G_S(\mathscr{O}_v) \to \Gamma_S(\mathscr{O}_v)$ is surjective for all $v \notin S$ gives an exact sequence of pointed sets

$$G(\mathbf{A}_k^S) \to \Gamma(\mathbf{A}_k^S) \xrightarrow{\delta} \coprod_{v \notin S} \mathrm{H}^1(k_v, G^0) \to \coprod_{v \notin S} \mathrm{H}^1(k_v, G).$$

To summarize, in terms of the diagram

$$\begin{array}{ccccccc}
G(k) & \longrightarrow & \Gamma(k) & \longrightarrow & \mathrm{H}^1(k, G^0) & \xrightarrow{j} & \mathrm{H}^1(k, G) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \theta_{S,G^0}} & & \\
G(\mathbf{A}_k^S) & \xrightarrow[\pi]{} & \Gamma(\mathbf{A}_k^S) & \xrightarrow[\delta]{} & \coprod_{v \notin S} \mathrm{H}^1(k_v, G^0) & &
\end{array}$$

with exact rows, we have

$$\ker f = j(\theta_{S,G^0}^{-1}(\delta(\Gamma(\mathbf{A}_k^S))))$$

inside of $\mathrm{H}^1(k, G)$. But as we have already noted below (6.2.1), $\pi : G(\mathbf{A}_k^S) \to \Gamma(\mathbf{A}_k^S)$ is surjective. Thus, $\delta(\Gamma(\mathbf{A}_k^S))$ is the distinguished point. That is, $\ker f = j(\ker \theta_{S,G^0}) = j(Ш^1_S(k, G^0))$. Finiteness of $Ш^1_S(k, G^0)$ is therefore sufficient to deduce finiteness of $\ker f$, as desired. This completes the reduction to the case when the k-smooth G is connected.

6.3 Reduction to the pseudo-reductive case

Now assume that $\theta_{S,G}$ is known to have finite fibers whenever G is a pseudo-reductive k-group and S is arbitrary. We shall use finiteness of class numbers (Theorem 1.3.1) to deduce the same for any smooth connected affine k-group G. The twisting method (as reviewed at the start of

§6.1) reduces our task to proving that $Ш^1_S(k,G)$ is finite for all smooth connected affine k-groups G. We may assume that S is non-empty; the set S of places will remain fixed for the remainder of this part of the argument. Whenever we speak of a *k-form* H' of a smooth affine k-group H, we mean a form for the étale topology on k (equivalently, $H'_K \simeq H_K$ as K-groups for some separable algebraic extension K/k).

The k-unipotent radical $\mathscr{R}_{u,k}(G)$ (i.e., the maximal smooth connected unipotent normal k-subgroup of G) gives rise to a quotient $Q = G/\mathscr{R}_{u,k}(G)$ that is pseudo-reductive over k. Since pseudo-reductivity is insensitive to passing to a k-form (for the étale topology!) of a smooth connected affine k-group, for every k-form Q' of Q the set $Ш^1_S(k,Q')$ is finite. Thus, to reduce to the pseudo-reductive case it suffices (by the twisting method) to prove in general that if

$$1 \to U \xrightarrow{j} G \xrightarrow{\pi} Q \to 1 \tag{6.3.1}$$

is a short exact sequence of smooth connected affine k-groups with U a unipotent k-subgroup of G that is stable under all automorphisms of G defined over separable extensions of k and if $Ш^1_S(k,Q')$ is finite for all k-forms Q' of Q then $Ш^1_S(k,G')$ is finite for all k-forms of G. The hypothesis that U is stable under G-automorphisms over *all* separable extensions of k presents no set-theoretic "largeness" problems because it suffices to check this using only finitely generated separable extensions. We can make an exact sequence analogous to (6.3.1) for any k-form of G (using suitable k-forms of U and Q).

We may assume U is nontrivial, as otherwise there is nothing to do. The finite-length derived series of the k-group U is stable under all automorphisms of G defined over separable extensions of k, and it has successive quotients that are commutative and connected, so by inducting on the length of the derived series of U we may assume that U is commutative (and nontrivial). It is enough to prove finiteness of $Ш^1_S(k,G)$, as all k-forms G' of G admit an exact sequence analogous to (6.3.1) (with a commutative left term). Since $Ш^1_S(k,Q)$ is finite, it suffices to prove that the map $Ш^1_S(k,G) \to Ш^1_S(k,Q)$ has finite fibers. Exactly as in our reduction to the connected case in §6.2, we can use a twisting argument (replacing G, U, and Q with compatible k-forms) to reduce to proving finiteness of the kernel of the map $Ш^1_S(k,G) \to Ш^1_S(k,Q)$.

Reasoning as in §6.2, $\ker(Ш^1_S(k,G) \to Ш^1_S(k,Q))$ is identified with $j(\theta_{S,U}^{-1}(\delta(Q(\mathbf{A}_k^S))))$, where

$$\delta : Q(\mathbf{A}_k^S) \to \coprod_{v \notin S} \mathrm{H}^1(k_v, U)$$

is the "direct sum" of connecting maps, but in contrast with §6.2 the map $\pi : G(\mathbf{A}_k^S) \to Q(\mathbf{A}_k^S)$ now merely has open and not necessarily full image. Since the normal k-subgroup $U \subseteq G$ is commutative, the right action of G on U through conjugation factors through a right action of Q on U. Using this action, [Se2, I, §5.5, Prop. 39] provides a natural right $Q(k)$-action on $\mathrm{H}^1(k,U)$ such that the orbits are the non-empty fibers of the map $j : \mathrm{H}^1(k,U) \to \mathrm{H}^1(k,G)$ and the orbit of the distinguished point is $\delta(Q(k))$. Also, the $\pi(G(k))$-action on the distinguished point is trivial. Similarly there is a right $Q(\mathbf{A}_k^S)$-action on $\coprod_{v \notin S} \mathrm{H}^1(k_v,U)$ such that its orbits are the non-empty fibers of the map

$$\coprod_{v \notin S} \mathrm{H}^1(k_v, U) \to \coprod_{v \notin S} \mathrm{H}^1(k_v, G)$$

and the orbit of the distinguished point is $\delta(Q(\mathbf{A}_k^S))$. By the construction it is clear that with respect to these actions, $\theta_{S,U}$ is equivariant via the group homomorphism $Q(k) \to Q(\mathbf{A}_k^S)$. Moreover, the $\pi(G(\mathbf{A}_k^S))$-action on the distinguished point $0_{S,U} \in \coprod_{v \notin S} \mathrm{H}^1(k_v,U)$ is trivial.

To prove finiteness of $j(\theta_{S,U}^{-1}(\delta(Q(\mathbf{A}_k^S))))$ it is equivalent to show that $\theta_{S,U}^{-1}(\delta(Q(\mathbf{A}_k^S)))$ is contained in finitely many $Q(k)$-orbits. But $\delta(Q(\mathbf{A}_k^S)) = 0_{S,U}.Q(\mathbf{A}_k^S)$ is a $Q(\mathbf{A}_k^S)$-orbit of a point on which $\pi(G(\mathbf{A}_k^S))$ acts trivially, so it is the set of translates of $0_{S,U}$ by a set of representatives for $\pi(G(\mathbf{A}_k^S))\backslash Q(\mathbf{A}_k^S)$. Equivariance of $\theta_{S,U}$ with respect to the group homomorphism $Q(k) \to Q(\mathbf{A}_k^S)$ therefore implies that $\theta_{S,U}^{-1}(\delta(Q(\mathbf{A}_k^S)))$ is the union of $Q(k)$-orbits of elements in the fibers $\theta_{S,U}^{-1}(0_{S,U}.q)$ as q ranges through a set of representatives for $\pi(G(\mathbf{A}_k^S))\backslash Q(\mathbf{A}_k^S)/Q(k)$. But this double coset space is finite because the open subgroup $\pi(G(\mathbf{A}_k^S))$ contains a compact open subgroup K and $K\backslash Q(\mathbf{A}_k^S)/Q(k)$ is finite (due to the finiteness of class numbers for Q, via Theorem 1.3.1). Hence, the problem is reduced to showing that $\theta_{S,U}$ has finite fibers for any smooth connected commutative unipotent k-group U. The twisting method reduces this to the finiteness of $Ш_S^1(k, U)$ for all smooth connected commutative unipotent k-groups U, and such finiteness in the commutative case was proved by Oesterlé [Oes, IV, 2.6(a)].

6.4 Application of structure of pseudo-reductive groups

By §6.3, it remains to prove that $Ш_S^1(k, G)$ is finite whenever G is a pseudo-reductive k-group and S is non-empty. The case of smooth connected commutative affine groups was settled by Oesterlé [Oes, IV, 2.6(a)], so we may and do assume G is non-commutative. By Theorem 2.3.6(ii) and Theorem 2.3.8, we may also assume that G is a generalized standard pseudo-reductive group.

Let $(G', k'/k, T', C)$ be the generalized standard presentation of G adapted to a choice of maximal k-torus T in G (see Definition 2.3.3 and Remark 2.3.4), so $C = Z_G(T)$ and there is a central extension

$$1 \to \mathrm{R}_{k'/k}(C') \to \mathrm{R}_{k'/k}(G') \rtimes C \to G \to 1 \tag{6.4.1}$$

in which k' is a nonzero finite reduced k-algebra, G' is a smooth affine k'-group whose fibers are absolutely pseudo-simple and either simply connected semisimple or basic exotic, and T' is a maximal k'-torus in G' whose centralizer is C'.

The 7-term exact sequence in pointed cohomology sets associated to a central extension of finite type k-group schemes is very well-known in the smooth case using Galois cohomology (see [Se2, I, §5.7]), and is reviewed from scratch in Appendix B.3 without smoothness conditions since this will be needed later. For now we only require smooth groups. More specifically, the central extension (6.4.1) provides a canonical connecting map of pointed sets

$$\Delta : \mathrm{H}^1(k, G) \to \mathrm{H}^2(k, \mathrm{R}_{k'/k}(C')), \tag{6.4.2}$$

and similarly with k_v-cohomologies. Thus, Δ induces a map

$$\Delta_{Ш} : Ш_S^1(k, G) \to Ш_S^2(k, \mathrm{R}_{k'/k}(C')) = \prod Ш_{S_i}^2(k_i', C_i'),$$

where $k' = \prod k_i'$ is the decomposition into factor fields, S_i' is the set of places of k_i' over S, C_i' is the k_i'-fiber of C', and $Ш_S^2$ denotes the kernel of the localization map away from S for H^2 on commutative group schemes of finite type.

Each C_i' is a Cartan k_i'-subgroup of G_i', and so is a torus when G_i' is semisimple. If instead G_i' is basic exotic, then we saw in the proof of Proposition 4.1.9 that there is a natural quotient map $C_i' \twoheadrightarrow \overline{C}_i'$ onto a k_i'-torus such that the induced map on $k_{i,s}'$-points is bijective. Thus, in the basic exotic cases there is an isomorphism $Ш_{S_i}^2(k_i', C_i') \simeq Ш_{S_i}^2(k_i', \overline{C}_i')$ onto the degree-2 Tate–Shafarevich group (relative to S_i') for a k_i'-torus. It follows that for all i, by [Oes, IV, 2.7(a)] (an application of Tate–Nakayama duality for tori) each $Ш_{S_i}^2(k_i', C_i')$ is finite. Hence, it suffices to prove that $\Delta_{Ш}$ has finite fibers.

Pick $c \in Ш^1_S(k, G)$ and consider $\Delta_{Ш}^{-1}(\Delta_{Ш}(c))$. If we choose a representative 1-cocycle $\gamma \in \mathrm{Z}^1(k_s/k, G) \subseteq G(k_s \otimes_k k_s)$ for c then we get an inner k-form G_γ of G. We likewise get a k-form E_γ of the middle term $E := \mathrm{R}_{k'/k}(G') \rtimes C$ in (6.4.1) by using the natural left G-action on E arising from conjugation and the central extension structure (6.4.1). Hence, we obtain a central extension

$$1 \to \mathrm{R}_{k'/k}(C') \to E_\gamma \to G_\gamma \to 1 \tag{6.4.3}$$

for a smooth connected affine k-group G_γ equipped with a bijection $t_{\gamma,k} : \mathrm{H}^1(k, G) \simeq \mathrm{H}^1(k, G_\gamma)$ that carries c to the distinguished point. By [Se2, I, §5.7, Prop. 4.4], this carries (6.4.2) to the connecting map $\delta_\gamma : Ш^1_S(k, G_\gamma) \to Ш^2_S(k, \mathrm{R}_{k'/k}(C'))$. Thus, it suffices to prove that $\ker \delta_\gamma$ is finite.

We cannot simply rename G_γ as G, since perhaps γ-twisting of (6.4.1) might not interact well with the chosen generalized standard presentation of G. The essential issue is to understand the effect of γ-twisting on the k-subgroup $\mathrm{R}_{k'/k}(G')$ in E. By [CGP, Prop. 8.1.2, Cor. A.7.11] this subgroup is its own derived group since the fibers of G' over the factor fields of k' are absolutely pseudo-simple and either simply connected semisimple or basic exotic. Hence, $\mathrm{R}_{k'/k}(G')$ is the derived group of E (as $E/\mathrm{R}_{k'/k}(G') = C$ is commutative). Thus, γ-twisting on E induces a (generally non-inner) twisting $\mathrm{R}_{k'/k}(G')_\gamma$ of the derived subgroup $\mathrm{R}_{k'/k}(G') = \mathscr{D}(E)$ of E and a twisting C_γ of the maximal commutative quotient $C = E/\mathscr{D}(E)$.

The commutative k-group C_γ is pseudo-reductive since it becomes isomorphic to C étale-locally over k. The k-group $\mathrm{R}_{k'/k}(G')_\gamma$ can be described as a Weil restriction:

Proposition 6.4.1. *Let k be an arbitrary field, k' a nonzero finite reduced k-algebra, and G' a smooth affine k'-group whose fiber over each factor field of k' is absolutely pseudo-simple and either simply connected semisimple or basic exotic. Let G be the smooth connected affine k-group $\mathrm{R}_{k'/k}(G')$.*

Any k-form H of G relative to the étale topology over k is k-isomorphic to $\mathrm{R}_{F'/k}(H')$ for a nonzero finite reduced k-algebra F' and a smooth affine F'-group H' whose fiber over each factor field of F' is absolutely pseudo-simple and either simply connected semisimple or basic exotic.

Proof. The k-group G is generalized standard, and its generalized standard presentation adapted to a choice of maximal k-torus T has the form $(G', k'/k, T', C)$ where $\phi : \mathrm{R}_{k'/k}(C') \to C$ is an isomorphism. Since the formation of generalized standard presentations is compatible with separable extension of the ground field, the property of ϕ being an isomorphism is independent of the choice of T (as it can be checked over k_s, and all maximal k_s-tori are $G(k_s)$-conjugate). The generalized standard property is insensitive to separable extension on k [CGP, Cor. 10.2.5], so all k-forms of G for the étale topology over k are generalized standard and satisfy the isomorphism property for ϕ in their generalized standard presentations. It follows that all such k-forms are k-isomorphic to a Weil restriction of the desired type. ■

By Proposition 6.4.1, the k-form $\mathrm{R}_{k'/k}(G')_\gamma = \mathscr{D}(E_\gamma)$ of $\mathrm{R}_{k'/k}(G') = \mathscr{D}(E)$ is k-isomorphic to $\mathrm{R}_{F'/k}(\mathscr{G}')$ for a nonzero finite reduced k-algebra F' and a smooth affine F'-group $\mathscr{G}'$ whose fiber over each factor field of F' is absolutely pseudo-simple and either simply connected semisimple or basic exotic. This underlies the proof the following lemma.

Lemma 6.4.2. *With notation as above, the natural map of sets $q : \mathrm{H}^1(k, E_\gamma) \to \mathrm{H}^1(k, C_\gamma)$ is injective.*

This lemma says that *all* non-empty fibers of q have one point, not just the fiber of q over the distinguished point of the target.

Proof. Choose $c \in \mathrm{H}^1(k, E_\gamma)$ and let γ' be a representative 1-cocycle for c. By γ'-twisting we get a k-form $\mathrm{R}_{F'/k}(\mathscr{G}')'$ of $\mathrm{R}_{F'/k}(\mathscr{G}')$ and an inner form E'_γ of E_γ fitting into a short exact sequence

$$1 \to \mathrm{R}_{F'/k}(\mathscr{G}')' \to E'_\gamma \to C'_\gamma \to 1$$

such that there is a bijection $\mathrm{H}^1(k, E_\gamma) \simeq \mathrm{H}^1(k, E'_\gamma)$ carrying the fiber $q^{-1}(q(c))$ over to the kernel of the map of pointed sets $\mathrm{H}^1(k, E'_\gamma) \to \mathrm{H}^1(k, C'_\gamma)$. Thus, it suffices to prove that $\mathrm{H}^1(k, \mathrm{R}_{F'/k}(\mathscr{G}')')$ is the trivial pointed set.

For each factor field F'_i of F', the fiber $\mathscr{G}'_i$ of $\mathscr{G}'$ over F'_i is absolutely pseudo-simple and either simply connected semisimple or basic exotic. Thus, by applying Proposition 6.4.1 to $(\mathscr{G}', F'/k)$, Lemma 4.1.6 and Theorem 5.1.1(i) yield the triviality of $\mathrm{H}^1(k, \mathrm{R}_{F'/k}(\mathscr{G}')')$ since k is a global function field. ■

Lemma 6.4.2 now reduces us to the following axiomatic finiteness problem (upon renaming G_γ as G and forgetting about pseudo-reductivity, which has served its purpose). Consider a central extension

$$1 \to \mathscr{C} \xrightarrow{j} E \xrightarrow{\pi} G \to 1 \tag{6.4.4}$$

of a smooth connected affine k-group G by a smooth connected commutative affine k-group $\mathscr{C}$. Assume that the abelianization map $E \to C := E/\mathscr{D}(E)$ induces an injective map of sets $\mathrm{H}^1(k, E) \to \mathrm{H}^1(k, C)$ and that $\mathscr{C} \simeq \mathrm{R}_{k'/k}(C')$ for a nonzero finite reduced k-algebra k' and a smooth commutative k'-group C' with connected fibers. We claim that the connecting map

$$\Delta_{\text{Ш}} : \text{Ш}^1_S(k, G) \to \text{Ш}^2_S(k, \mathscr{C})$$

has finite kernel. Applying this to (6.4.3) (thanks to Lemma 6.4.2) would then complete the proof of Theorem 1.3.3.

An easy diagram chase gives that $\ker \Delta_{\text{Ш}}$ is the image by $\pi : \mathrm{H}^1(k, E) \to \mathrm{H}^1(k, G)$ of the set of elements $x \in \mathrm{H}^1(k, E)$ such that the element $\theta_{S,E}(x) \in \coprod_{v \notin S} \mathrm{H}^1(k_v, E)$ is in the image of $\coprod_{v \notin S} \mathrm{H}^1(k_v, \mathscr{C}) = \bigoplus_{v \notin S} \mathrm{H}^1(k_v, \mathscr{C})$ under j. In other words,

$$\ker \Delta_{\text{Ш}} = \pi(\theta_{S,E}^{-1}(j(\bigoplus_{v \notin S} \mathrm{H}^1(k_v, \mathscr{C})))).$$

Let $f : \mathscr{C} \to C$ denote the composition of $j : \mathscr{C} \to E$ and the quotient map $E \twoheadrightarrow C$. Using the assumed injectivity of the map of sets $\mathrm{H}^1(k, E) \to \mathrm{H}^1(k, C)$, the k-group map $E \twoheadrightarrow C$ thereby induces an injective map

$$\theta_{S,E}^{-1}(j(\bigoplus_{v \notin S} \mathrm{H}^1(k_v, \mathscr{C}))) \hookrightarrow \theta_{S,C}^{-1}(f(\bigoplus_{v \notin S} \mathrm{H}^1(k_v, \mathscr{C}))).$$

The centrality of the given extension structure (6.4.4) implies that $\mathrm{H}^1(k, \mathscr{C})$ naturally acts on $\mathrm{H}^1(k, E)$ with orbits that are the non-empty fibers of $\pi : \mathrm{H}^1(k, E) \to \mathrm{H}^1(k, G)$. Moreover, by the method of construction, the natural map $\mathrm{H}^1(k, E) \hookrightarrow \mathrm{H}^1(k, C)$ is $\mathrm{H}^1(k, \mathscr{C})$-equivariant with respect to the natural additive translation of $\mathrm{H}^1(k, \mathscr{C})$ on $\mathrm{H}^1(k, C)$ via $\mathrm{H}^1(f)$. Our finiteness problem is to show that $\theta_{S,E}^{-1}(j(\bigoplus_{v \notin S} \mathrm{H}^1(k_v, \mathscr{C})))$ is contained in finitely many $\mathrm{H}^1(k, \mathscr{C})$-orbits on $\mathrm{H}^1(k, E)$, so it suffices to prove that $\theta_{S,C}^{-1}(f(\bigoplus_{v \notin S} \mathrm{H}^1(k_v, \mathscr{C})))$ is contained in finitely many $\mathrm{H}^1(k, \mathscr{C})$-orbits on $\mathrm{H}^1(k, C)$.

In terms of the commutative diagram of *abelian groups*

$$\begin{array}{ccc} \mathrm{H}^1(k,\mathscr{C}) & \xrightarrow{\mathrm{H}^1(f)} & \mathrm{H}^1(k,C) \\ \downarrow{\scriptstyle \theta_{S,\mathscr{C}}} & & \downarrow{\scriptstyle \theta_{S,C}} \\ \bigoplus_{v\notin S}\mathrm{H}^1(k_v,\mathscr{C}) & \longrightarrow & \bigoplus_{v\notin S}\mathrm{H}^1(k_v,C) \end{array}$$

we have to show that the $\theta_{S,C}$-preimage of the image along the bottom side has finite image in the cokernel along the top side. Since $\mathscr{C} \simeq \mathrm{R}_{k'/k}(C')$, in terms of the factor fields k'_i of k' and the k'_i-fiber C'_i of C' we have $\mathrm{H}^1(k,\mathscr{C}) \simeq \prod_i \mathrm{H}^1(k'_i, C'_i)$ and similarly $\mathrm{H}^1(k_v,\mathscr{C}) \simeq \prod_i(\prod_{w_i}\mathrm{H}^1(k'_{i,w_i}, C'_i))$ for each $v \notin S$, with w_i ranging through the places of k'_i over v (Lemma 4.1.6). Hence, $\theta_{S,\mathscr{C}}$ is identified with the product map $\prod_i \theta_{S'_i,C'_i}$ where S'_i is the set of places of k'_i over S. Each map $\theta_{S'_i,C'_i}$ has finite cokernel by [Oes, IV, 2.6(b)], so $\theta_{S,\mathscr{C}}$ has finite cokernel. Moreover, $\theta_{S,C}$ has finite fibers since C is commutative, so the desired result is now obvious. This completes the proof of Theorem 1.3.3.

7. Applications

Our finiteness results for class numbers and Tate–Shafarevich sets in the affine case have interesting consequences for finiteness properties of cohomology of group schemes over rings of S-integers of global function fields, as well as over proper curves over finite fields. This rests on some additional finiteness results in the local case, so we begin with the latter before turning to global applications.

7.1 Cohomological finiteness over local function fields

For what follows it will be convenient to first recall a few general facts concerning smooth connected unipotent groups over imperfect fields. Although a quotient of a k-split smooth connected unipotent k-group is always k-split, we noted in §1.7 that smooth connected k-subgroups can fail to be k-split even in the commutative case. The following notion for unipotent groups is analogous to anisotropicity for tori:

Definition 7.1.1. A smooth connected unipotent group U over a field k is *k-wound* if there are no nonconstant k-morphisms to U from the affine k-line (as k-schemes).

If k is perfect then the only k-wound U is the trivial k-group. By [CGP, Thm. B.3.4], for any smooth connected unipotent k-group U there is a unique maximal k-split smooth connected k-subgroup $U_{\mathrm{split}} \subseteq U$ and it enjoys the following properties: it is normal in U, the quotient U/U_{split} is k-wound, the formation of U_{split} commutes with separable extension on k, and there are no nontrivial k-homomorphisms $U' \to U$ when U' is k-split and U is k-wound.

Proposition 7.1.2. *Let K/k be a finite separable extension of non-archimedean local fields and let G be a smooth connected affine k-group. The fibers of the restriction map $\mathrm{H}^1(k,G) \to \mathrm{H}^1(K,G)$ are finite.*

This result is only interesting when $\mathrm{char}(k) > 0$, since otherwise $\mathrm{H}^1(k,G)$ is finite. In [CGP, Ex. 11.3.3] there are examples (over any local function field k) of commutative pseudo-reductive k-groups C for which $\mathrm{H}^1(k,C)$ is infinite.

Proof. By the étale twisting method it is equivalent to prove in general that the kernel of the restriction map in cohomology is finite. Grant the pseudo-reductive case for a moment. In general there is a unique exact sequence of smooth connected affine k-groups

$$1 \to U \to G \xrightarrow{\pi} G' \to 1$$

with unipotent U and a pseudo-reductive k-group G'. By the twisting method (for the étale topology over k), which preserves pseudo-reductivity, it suffices to show that the kernel of $\mathrm{H}^1(k, G) \to \mathrm{H}^1(K, G)$ meets the image of $\mathrm{H}^1(k, U) \to \mathrm{H}^1(k, G)$ in a finite set. The open map $\pi : G(K) \to G'(K)$ has image with finite index, by Proposition 3.1.3 and Proposition 4.1.9. The connecting map $\delta : G'(K) \to \mathrm{H}^1(K, U)$ carries any $g' \in G'(K)$ to the K-isomorphism class of the right U-torsor $\pi^{-1}(g')$, so $\delta(g')$ only depends on g' modulo left multiplication by $\pi(G(K))$. Hence, δ factors through the finite set $\pi(G(K))\backslash G'(K)$ and thus has finite image. It suffices to show that the finite set $\delta(G'(K))$ has finite preimage in $\mathrm{H}^1(k, U)$, so we have reduced the general problem to two cases: pseudo-reductive k-groups and unipotent k-groups.

Consider the unipotent case. The case of k-split U is trivial, so we can assume $\mathrm{char}(k) > 0$. The formation of the maximal k-split smooth connected unipotent normal k-subgroup $\mathscr{R}_{us,k}(U)$ of U is étale-local on k [CGP, Thm. B.3.4] and the degree-1 Galois cohomology of $\mathscr{R}_{us,k}(U)$ vanishes, so by twisting we see that $\mathrm{H}^1(k, U) \to \mathrm{H}^1(k, U/\mathscr{R}_{us,k}(U))$ is injective. The same holds over K, so we may replace U with $U/\mathscr{R}_{us,k}(U)$ to reduce to the k-wound unipotent case. In this case there is a composition series whose successive quotients are commutative and k-wound [CGP, Prop. B.3.2], so we reduce to the commutative k-wound case.

For a commutative k-wound U, the restriction map of interest is the degree-1 cohomology map over k induced by the inclusion $j : U \to \mathrm{R}_{K/k}(U_K)$. Let $U' = \mathrm{coker}\, j$. The kernel of $\mathrm{H}^1(j)$ is identified with the cokernel of the map $\mathrm{R}_{K/k}(U_K)(k) \to U'(k)$ whose image is open (by smoothness). It is therefore enough to show that U' is k-wound, as then $U'(k)$ is compact by [Oes, VI, §1]. Rather more generally, if k is an arbitrary field, K is a nonzero finite étale k-algebra, and U is a commutative k-wound smooth connected unipotent k-group then we claim that $\mathrm{R}_{K/k}(U_K)/U$ is k-wound. It suffices to treat the case when k is separably closed since the property of being wound (or not) is insensitive to separable algebraic extension of the ground field, and then the result is obvious since K is a product of copies of k (so $\mathrm{R}_{K/k}(U_K)$ is a power of U in which U is diagonally embedded via j).

Now we treat the pseudo-reductive case. First assume G is commutative. If T is the maximal k-torus in G then the unipotent quotient G/T is k-wound by Lemma 4.1.4, so $(G/T)_K$ is K-wound (as K/k is separable). Thus, $(G/T)(K)$ is compact, so the open image of $G(K) \to (G/T)(K)$ has finite index. By a simple diagram chase with commutative cohomology, the finiteness problem for G is reduced to the analogous problems for G/T (which was already settled) and for T (which is immediate from Proposition 4.1.7(i)).

In the non-commutative pseudo-reductive case, by Theorem 2.3.6(ii) and Theorem 2.3.8 it suffices to treat non-commutative generalized standard pseudo-reductive k-groups G. The twisting method (which preserves generalized standardness, due to [CGP, Cor. 10.2.5]) reduces the problem to proving that $\mathrm{H}^1(k, G) \to \mathrm{H}^1(K, G)$ has finite kernel. Choose a maximal k-torus T in G and let $(G', k'/k, T', C)$ be the generalized standard presentation of G adapted to T (Remark 2.3.4), so $C = Z_G(T)$ and there is a central extension

$$1 \to \mathrm{R}_{k'/k}(C') \to \mathrm{R}_{k'/k}(G') \rtimes C \to G \to 1$$

with k' a nonzero finite reduced k-algebra, G' a smooth affine k'-group whose fibers are absolutely pseudo-simple and either simply connected semisimple or basic exotic, and C a commutative pseudo-reductive k-group. For $E := \mathrm{R}_{k'/k}(G') \rtimes C$ and a G-valued 1-cocycle γ for the étale topology on $\mathrm{Spec}(k)$, the injectivity in Lemma 6.4.2 (which applies just as well for K-cohomology) gives that $\mathrm{H}^1(k, E_\gamma) \to \mathrm{H}^1(K, E_\gamma)$ has finite fibers since the same holds for C in place of E by the settled commutative pseudo-reductive case. Since $\mathrm{R}_{k'/k}(C')_K = \mathrm{R}_{K'/K}(C'_{K'})$ for $K' = k' \otimes_k K$, for all m we have $\mathrm{H}^m(K, \mathrm{R}_{k'/k}(C')) = \mathrm{H}^m(k' \otimes_k K, C')$ (Lemma 4.1.6). This is identified with $\mathrm{H}^m(k' \otimes_k K, \mathscr{T})$ for a suitable k'-torus $\mathscr{T}$ (via the argument with Theorem 2.3.8(ii) used in the proof of Proposition 4.1.9), so by Proposition 4.1.7(i) it is finite for $m = 1$. Also, $\mathrm{R}_{k'/k}(C')$ is invariant under γ-twisting for any γ (by centrality). Thus, by a straightforward diagram chase with the twisting method and its compatibility with the connecting map $\mathrm{H}^1 \to \mathrm{H}^2$ in the smooth case [Se2, I, §5.7, Prop. 44], the finiteness of the kernel of $\mathrm{H}^1(k, G) \to \mathrm{H}^1(K, G)$ is reduced to the finiteness of the kernel of the analogous restriction map for degree-2 cohomology of $\mathrm{R}_{k'/k}(C')$.

The composite isomorphisms

$$\mathrm{H}^2(k, \mathrm{R}_{k'/k}(C')) \simeq \mathrm{H}^2(k', C') \simeq \mathrm{H}^2(k', \mathscr{T})$$

and

$$\mathrm{H}^2(K, \mathrm{R}_{k'/k}(C')) \simeq \mathrm{H}^2(k' \otimes_k K, C') \simeq \mathrm{H}^2(k' \otimes_k K, \mathscr{T})$$

are compatible with the evident restriction maps in Galois cohomology over the factor fields. Thus, it suffices to prove that for any finite separable extension K/k of non-archimedean local fields and any k-torus T, the restriction map $\mathrm{H}^2(k, T) \to \mathrm{H}^2(K, T)$ has finite kernel. If $n = [K : k]$ then this kernel is contained in $\mathrm{H}^2(k, T)[n]$. Such n-torsion is the image of $\mathrm{H}^2(k, T[n])$ (using *fppf* cohomology in case $\mathrm{char}(k)|n$), and this latter H^2 is finite by Proposition 4.1.7(ii). ■

There is an interesting refinement concerning finiteness for the Galois cohomology of pseudo-reductive groups over local function fields (cf. [Se2, III, §4.3, Rem. (2)]):

Proposition 7.1.3. *Let k be a local function field. If G is a pseudo-reductive k-group that is generated by its maximal k-tori then $\mathrm{H}^1(k, G)$ is finite.*

The torus hypothesis on G is satisfied when $G = \mathscr{D}(G)$ [CGP, Prop. A.2.11] and this hypothesis cannot be removed: for any local function field k, [CGP, Ex. 11.3.3] provides examples of non-reductive commutative pseudo-reductive k-groups C such that $\mathrm{H}^1(k, C)$ is infinite.

Proof. If G is commutative then it is a k-torus due to the hypotheses, so the commutative case follows from Lemma 4.1.7(i). Now we may and do assume that G is non-commutative. By Theorem 2.3.6(ii) and Theorem 2.3.8, we may assume that G is a generalized standard pseudo-reductive k-group. In particular, $\mathscr{D}(G)$ is generalized standard [CGP, Prop. 10.2.3]. The quotient $G/\mathscr{D}(G)$ is commutative and generated by k-tori, so it is a k-torus. Thus, any maximal k-torus T in G maps onto $G/\mathscr{D}(G)$. Letting Z denote the maximal central k-torus in G, the multiplication map $Z \times (T \cap \mathscr{D}(G)) \to T$ is surjective with finite kernel [CGP, Lemma 1.2.5(ii)], so $Z \to G/\mathscr{D}(G)$ is surjective. Hence, there is a central extension

$$1 \to \mu \to \mathscr{D}(G) \times Z \to G \to 1$$

with $\mu = Z \cap \mathscr{D}(G) = Z \cap (T \cap \mathscr{D}(G))$ a *finite* k-group of multiplicative type.

Using the finiteness of $\mathrm{H}^2(k, \mu)$ (Proposition 4.1.7(ii)), the twisting method (which preserves generalized standardness [CGP, Cor. 10.2.5]) reduces the finiteness of $\mathrm{H}^1(k, G)$ to the finiteness of $\mathrm{H}^1(k, \mathscr{D}(G))$ since $\mathrm{H}^1(k, \cdot)$ is finite on k-tori (Proposition 4.1.7(i)). We may therefore replace G with $\mathscr{D}(G)$, so now G is also perfect [CGP, Prop. 1.2.6]. Of course, we can also assume $G \neq 1$.

Since G is a nontrivial perfect generalized standard pseudo-reductive k-group, for a generalized standard presentation $(G', k'/k, T', C)$ the associated k-homomorphism $\mathrm{R}_{k'/k}(G') \to G$

with central kernel and normal image is surjective (the cokernel is perfect, yet is also a quotient of the commutative C). The compatibility of $\mathrm{R}_{k'/k}$ with the formation of the scheme-theoretic center of smooth affine groups [CGP, Prop. A.5.15(1)] implies that $G \simeq \mathrm{R}_{k'/k}(G')/Z$ for some k-subgroup scheme $Z \subseteq \mathrm{R}_{k'/k}(Z_{G'})$. The pseudo-reductivity of such a quotient G is equivalent to the pseudo-reductivity of $\mathrm{R}_{k'/k}(C')/Z$ for the commutative Cartan k'-subgroup $C' = Z_{G'}(T')$ in G', but we will not use this property; we shall prove the finiteness of $\mathrm{H}^1(k, \mathrm{R}_{k'/k}(G')/Z)$ for any non-archimedean local field k, nonzero finite reduced k-algebra k', smooth affine k'-group G' with absolutely pseudo-simple fibers that are either simply connected semisimple or basic exotic, and k-subgroup scheme $Z \subseteq \mathrm{R}_{k'/k}(Z_{G'})$.

Consider the central extension

$$1 \to Z \to \mathrm{R}_{k'/k}(G') \to G \to 1$$

and the connecting map $\delta : \mathrm{H}^1(k, G) \to \mathrm{H}^2(k, Z)$ (using *fppf* cohomology and Appendix B.3 in case Z is not smooth). Let us prove that this map is injective. By Proposition B.3.3(i), under twisting by a 1-cocycle $\gamma \in \mathrm{Z}^1(k_s/k, G)$ the resulting bijection $t_{\gamma,k} : \mathrm{H}^1(k, G) \simeq \mathrm{H}^1(k, G_\gamma)$ of sets carries δ over to the connecting map δ_γ arising from the γ-twisted central extension

$$1 \to Z \to \mathrm{R}_{k'/k}(G')_\gamma \to G_\gamma \to 1. \tag{7.1.1}$$

Injectivity of δ is now reduced to the triviality of $\ker \delta_\gamma$ for all γ. For this it suffices to prove that the middle term in (7.1.1) has vanishing degree-1 cohomology. But $\mathrm{R}_{k'/k}(G')_\gamma \simeq \mathrm{R}_{k'_1/k}(G'_1)$ for another pair $(G'_1, k'_1/k)$ that depends on γ (by Proposition 6.4.1), so $\mathrm{H}^1(k, \mathrm{R}_{k'/k}(G')_\gamma) = 1$ by Lemma 4.1.6 and Theorem 5.1.1(i). Since δ is now proved to be injective, it suffices to prove that $\mathrm{H}^2(k, Z)$ is finite.

There is a unique finite k-subgroup M in Z of multiplicative type such that Z/M is unipotent. Indeed, uniqueness is clear and for existence it suffices to treat the case $Z = \mathrm{R}_{k'/k}(Z_{G'}) = \prod \mathrm{R}_{k'_i/k}(Z_{G'_i})$ where $\{k'_i\}$ is the set of factor fields of k' and G'_i denotes the k'_i-fiber of G'. Each G'_i is either simply connected semisimple or basic exotic. If G'_i is semisimple then the k'_i-fiber $Z_{G'_i}$ is a finite k'_i-group of multiplicative type, and by [CGP, Cor. 7.2.5(2)] the same holds in all basic exotic cases except for when $\mathrm{char}(k) = 2$ and $(G'_i)^{\mathrm{ss}}_{\overline{k}'_i}$ is of type C_n with even n, in which case $Z_{G'_i} = \mathrm{R}_{k''_i/k'_i}(\mu_2)$ for $k''_i = {k'_i}^{1/2}$. Thus, in all cases $Z := \mathrm{R}_{k'/k}(Z_{G'}) = \mathrm{R}_{k''/k}(\mu'')$ for a nonzero finite reduced k-algebra k'' and a finite k''-group μ'' of multiplicative type. Hence, the existence of M in Z is clear over a sufficiently large finite Galois extension F/k such that the Cartier dual of μ'' has constant fibers over $F \otimes_k k''$. Uniqueness and Galois descent imply existence over k.

It now suffices to prove that if C is any commutative affine k-group scheme of finite type containing a finite multiplicative k-subgroup M such that $U := C/M$ is unipotent then $\mathrm{H}^2(k, C)$ is finite. Since $\mathrm{H}^2(k, M)$ is finite (Proposition 4.1.7(ii)), it suffices to prove that $\mathrm{H}^2(k, U) = 0$ for commutative unipotent k-group schemes U. By [SGA3, $\mathrm{VII_A}$, 8.3], there is an infinitesimal k-subgroup $U_0 \subseteq U$ such that U/U_0 is smooth (but possibly disconnected). Hence, it suffices to separately treat the cases when U is finite or U is smooth and connected. By using a composition series provided by [SGA3, XVII, Thm. 3.5], the case of finite U is reduced to the cases when $U = \alpha_p$ or U is a finite étale p-torsion k-group. In both of these cases, as well as in the smooth connected case, there is a finite extension field k'/k such that $U_{k'}$ is a k'-subgroup of a k'-split smooth connected commutative unipotent k'-group U'. The k-group embedding $U \hookrightarrow \mathrm{R}_{k'/k}(U_{k'})$ realizes U as a k-subgroup of the k-split smooth connected commutative unipotent k-group $\mathrm{R}_{k'/k}(U')$. The smooth connected unipotent quotient $Q = \mathrm{R}_{k'/k}(U')/U$ is k-split since $\mathrm{R}_{k'/k}(U')$ is k-split, so the vanishing of $\mathrm{H}^2(k, U)$ is reduced to the vanishing of $\mathrm{H}^1(k, Q)$ and $\mathrm{H}^2(k, \mathrm{R}_{k'/k}(U'))$.

Since Q and $\mathrm{R}_{k'/k}(U')$ are k-split, it remains to prove the vanishing of $\mathrm{H}^i(k, \mathbf{G}_{\mathrm{a}})$ for all $i > 0$. It is a classical fact that even $\mathrm{H}^i(K/k, K) = 0$ for $i > 0$ and any finite Galois extension K/k, since K is an "induced" $\mathrm{Gal}(K/k)$-module due to the normal basis theorem. ■

7.2 Finiteness with integrality conditions

In the study of arithmetic groups it is natural to consider integral structures on groups over local and global fields. Over rings of S-integers and their completions in characteristic 0, some interesting finiteness results for the cohomology of affine group schemes of finite type were proved in [Nis] and [GM, §1–§6]. It is explained in [GM, §7] how to prove weaker analogous results in the function field case. The key missing ingredient for proving such results in full strength in the function field case was Theorem 1.3.3(i), so now the method of proof of [GM, Prop. 5.1] works *verbatim* to establish the following analogous result:

Proposition 7.2.1. *Let k be a global function field and S a finite non-empty set of places of k. Let G be an affine $\mathscr{O}_{k,S}$-group scheme of finite type with smooth connected generic fiber. The set $\mathrm{H}^1(\mathscr{O}_{k,S}, G)$ of isomorphism classes of right G-torsors over $\mathscr{O}_{k,S}$ for the fppf topology is finite.*

The hypotheses on the generic fiber are necessary, as is seen by the examples $G = \mu_p$ and $G = \mathbf{Z}/p\mathbf{Z}$ with $p = \mathrm{char}(k) > 0$. By using Proposition 7.2.1, the proof of the main result in [GM] (i.e., [GM, Thm. 1.1]) in characteristic 0 carries over to the global function field case provided that we impose smoothness and connectedness conditions over k:

Theorem 7.2.2. *Let k be a global function field and S a finite non-empty set of places of k. Let X be a flat $\mathscr{O}_{k,S}$-scheme of finite type equipped with an action by an affine $\mathscr{O}_{k,S}$-group scheme G of finite type. For each $v \notin S$, let $\overline{\mathscr{O}}_v$ denote the valuation ring of an algebraic closure of k_v.*

Let $Z_0 \subseteq X$ be an $\mathscr{O}_{k,S}$-flat closed subscheme such that the (representable) G_k-stabilizer of $(Z_0)_k$ in X_k is smooth and connected. The set of closed subschemes $Z \subseteq X$ such that $Z \otimes \overline{\mathscr{O}}_v$ is $G(\overline{\mathscr{O}}_v)$-conjugate to $Z_0 \otimes \overline{\mathscr{O}}_v$ for all $v \notin S$ consists of finitely many $G(\mathscr{O}_{k,S})$-orbits.

This result improves on [GM, Thm. 7.7] by eliminating hypotheses on unipotent radicals. An interesting nontrivial case of this theorem is $Z_0 \in X(\mathscr{O}_{k,S})$ with generic point in $X(k)$ having smooth connected G_k-stabilizer; this is an "integral" analogue of Theorem 1.3.3(ii).

7.3 The case $S = \emptyset$

Let k be a global function field, and let G be an affine k-group scheme of finite type. Although finiteness of class numbers for G requires working with a finite non-empty set S of places of k, we can prove finiteness results even when S is empty. Let U be an open subgroup of $G(\mathbf{A}_k)$ and consider the double coset space $G(k)\backslash G(\mathbf{A}_k)/U$. Without any further hypothesis on U this is generally not finite: if U is compact then such finiteness amounts to the compactness of $G(k)\backslash G(\mathbf{A}_k)$, which fails when G is a k-isotropic connected semisimple k-group (Theorem 5.1.1(ii)). However, if U is large enough then we do have a finiteness result, as follows.

Theorem 7.3.1. *Let k be a global function field and G a smooth connected affine k-group. Let T be a maximal k-split torus in G, and let U be an open subgroup of $G(\mathbf{A}_k)$. Define $T(\mathbf{A}_k)^1 \subseteq T(\mathbf{A}_k)$ as in Definition 4.2.2. If $U \cap T(\mathbf{A}_k)$ has finite-index image in the $\mathbf{Z}$-lattice $T(\mathbf{A}_k)/T(\mathbf{A}_K)^1$ then $G(k)\backslash G(\mathbf{A}_k)/U$ is finite. In particular, if G does not contain GL_1 as a k-subgroup then $G(k)\backslash G(\mathbf{A}_k)$ is compact.*

Note that all choices of T are $G(k)$-conjugate [CGP, Thm. C.2.3], but the hypothesis on U is sensitive to the choice of T.

Proof. The strategy is to revisit the proof in §5 of finiteness of class numbers over function fields (Theorem 1.3.1) and adapt those arguments to work for the (typically non-compact) open subgroup U in place of the preimage in $G(\mathbf{A}_k)$ of a compact open subgroup of $G(\mathbf{A}_k^S)$.

Step 1. Suppose G is commutative. The quotient $\overline{G} = G/T$ has an anisotropic maximal k-torus, hence no nontrivial k-rational characters, so the coset space $\overline{G}(k)\backslash\overline{G}(\mathbf{A}_k)$ is compact [Oes, IV, 1.3]. Due to the cohomological triviality properties of T, the short exact sequence of k-groups

$$1 \to T \to G \to \overline{G} \to 1$$

induces exact sequences on k-points and $\mathbf{A}_k$-points. Moreover, $G(\mathbf{A}_k) \to \overline{G}(\mathbf{A}_k)$ is an open map since T is smooth and connected. For the open image $\overline{U}$ of U under this map, $\overline{G}(k)\backslash\overline{G}(\mathbf{A}_k)/\overline{U}$ is finite due to the compactness of $\overline{G}(k)\backslash\overline{G}(\mathbf{A}_k)$. Thus, it suffices to prove finiteness of the fibers of the map

$$G(k)\backslash G(\mathbf{A}_k)/U \to \overline{G}(k)\backslash\overline{G}(\mathbf{A}_k)/\overline{U}.$$

This map is a homomorphism since G is commutative, and image of $T(k)\backslash T(\mathbf{A}_k)/(U\cap T(\mathbf{A}_k))$ is its kernel, so it suffices to prove finiteness of this latter double coset space. By the compactness of $T(k)\backslash T(\mathbf{A}_k)^1$, the set $T(k)\backslash T(\mathbf{A}_k)^1/(U\cap T(\mathbf{A}_k)^1)$ is finite and passing to the quotient by this yields the quotient of $T(\mathbf{A}_k)/T(\mathbf{A}_k)^1$ modulo the image of $U\cap T(\mathbf{A}_k)$. By hypothesis this latter quotient is finite.

Step 2. Next, we show that it suffices to prove the result for the pseudo-reductive quotient $\overline{G} := G/\mathscr{R}_{u,k}(G)$. The connectedness of $\mathscr{R}_{u,k}(G)$ ensures that the diagram of smooth affine k-groups

$$1 \to \mathscr{R}_{u,k}(G) \to G \to \overline{G} \to 1$$

induces an open map $G(\mathbf{A}_k) \to \overline{G}(\mathbf{A}_k)$, so the image $\overline{U}$ of U in $\overline{G}(\mathbf{A}_k)$ is an open subgroup of $\overline{G}(\mathbf{A}_k)$. By Proposition 3.1.3, the k-isomorphic image $\overline{T}$ of T in $\overline{G}$ is a maximal k-split torus in $\overline{G}$.

The identification of $\overline{T}$ with T carries $U\cap T(\mathbf{A}_k)$ into $\overline{U}\cap\overline{T}(\mathbf{A}_k)$, so $\overline{U}$ satisfies the same initial hypotheses with respect to $(\overline{G},\overline{T})$ as U does with respect to (G,T). Hence, granting the pseudo-reductive case, $\overline{G}(k)\backslash\overline{G}(\mathbf{A}_k)/\overline{U}$ is finite. The intersection of $\overline{G}(k)$ with the image of $G(\mathbf{A}_k) \to \overline{G}(\mathbf{A}_k)$ contains the image of $G(k)$ with finite index due to [Se2, I, §5.5, Prop. 39] and the finiteness of $Ш^1_\emptyset(k,\mathscr{R}_{u,k}(G))$ [Oes, IV, 2.6(a)], so the fiber of the map

$$G(k)\backslash G(\mathbf{A}_k)/U \to \overline{G}(k)\backslash\overline{G}(\mathbf{A}_k)/\overline{U}$$

through the double coset of a fixed $g \in G(\mathbf{A}_k)$ consists of points represented by ug_i for a finite set of elements $g_i \in G(\mathbf{A}_k)$ and arbitrary $u \in \mathscr{R}_{u,k}(G)(\mathbf{A}_k)$. But u only matters modulo left multiplication by $\mathscr{R}_{u,k}(G)(k)$, so compactness of $\mathscr{R}_{u,k}(G)(k)\backslash\mathscr{R}_{u,k}(G)(\mathbf{A}_k)$ [Oes, IV, 1.3] completes the reduction to the case when G is pseudo-reductive.

Step 3. From now on G is pseudo-reductive over k, and non-commutative (by Step 1). By Theorem 2.3.6(ii) and Theorem 2.3.8, we can assume that G is a generalized standard pseudo-reductive k-group. (When applying Theorem 2.3.6(ii) we also use Lemma 4.2.4.) Using notation from §5.2 that rests on a choice of generalized standard presentation of G (see Definition 2.3.3 and Remark 2.3.4), there is a pair of exact sequences of k-groups

$$1 \to \mathrm{R}_{k'/k}(G') \to \mathscr{E} \to (C\times\mathscr{Z})/Z \to 1$$

and

$$1 \to \mathscr{Z} \to \mathscr{E} \to G \to 1$$

that are respectively (5.2.2) and (5.2.3).

Let $\mathscr{U}$ denote the open preimage of U in $\mathscr{E}(\mathbf{A}_k)$, so $\mathscr{U}$ maps onto U due to the cohomological triviality properties of $\mathscr{Z}$. The smooth connected preimage of T in $\mathscr{E}$ contains a maximal k-split k-torus $\mathscr{T}$ of $\mathscr{E}$ (Proposition 3.1.3), and $\mathscr{T}$ maps onto T. Since $(\mathscr{T}\cap\mathscr{Z})^0_{\rm red}$ is the unique maximal k-split k-torus of $\mathscr{Z}$, $\mathscr{U}\cap\mathscr{T}(\mathbf{A}_k)$ has finite-index image in $\mathscr{T}(\mathbf{A}_k)/\mathscr{T}(\mathbf{A}_k)^1$. But $\mathscr{E}(k)\backslash\mathscr{E}(\mathbf{A}_k)/\mathscr{U}\to G(k)\backslash G(\mathbf{A}_k)/U$ is surjective (due to the cohomological triviality properties of $\mathscr{Z}$), so we may therefore replace (G,T) with $(\mathscr{E},\mathscr{T})$ to reduce to the case when there is an exact sequence of smooth connected affine k-groups

$$1\to \mathrm{R}_{k'/k}(G')\to G\to G''\to 1 \tag{7.3.1}$$

with a nonzero finite reduced k-algebra k', a smooth affine k'-group G' whose fibers are absolutely pseudo-simple and either simply connected semisimple or basic exotic, and a smooth connected affine k-group G'' for which the desired finiteness result holds (such as commutative G'').

Consider the decomposition $k'=\prod k'_i$ into a finite product of fields. Let G'_i be the fiber of G' over k'_i, so $\mathrm{R}_{k'/k}(G')=\prod \mathrm{R}_{k'_i/k}(G'_i)$. We need the following general claim concerning (7.3.1):

Lemma 7.3.2. *Each factor* $\mathrm{R}_{k'_i/k}(G'_i)$ *is normal in* G.

Proof. By Galois descent we may make a preliminary finite Galois extension on k so that each k'_i/k is purely inseparable, and hence each factor ring $k'_{i,s}:=k'_i\otimes_k k_s$ of $k'_s:=k'\otimes_k k_s$ is a field. Thus, conjugation on G_{k_s} by any $g\in G(k_s)$ permutes the finite set of k_s-subgroups $\mathrm{R}_{k'_i/k}(G'_i)_{k_s}=\mathrm{R}_{k'_{i,s}/k_s}(G'_{i,k'_{i,s}})$ since the pair $(G',k'/k)_{k_s}=(G'_{k'_s},k'_s/k_s)$ is functorial with respect to k_s-isomorphisms in the k_s-group $\mathrm{R}_{k'/k}(G')_{k_s}=\mathrm{R}_{k'_s/k_s}(G'_{k_s})$ due to (the proof of) [CGP, Prop. 10.2.4]. Hence, each $\mathrm{R}_{k'_i/k}(G'_i)_{k_s}$ is normalized by a finite-index subgroup of $G(k_s)$. But every finite-index subgroup of $G(k_s)$ is Zariski-dense in G_{k_s} (since the Zariski closure H of such a subgroup is smooth, so $(G/H)(k_s)=G(k_s)/H(k_s)$ yet G_{k_s}/H is smooth and connected). Thus, each $\mathrm{R}_{k'_i/k}(G'_i)$ is normal in G because such normality holds over k_s. ∎

We may now form the exact sequence

$$1\to \mathrm{R}_{k'/k}(G')/\,\mathrm{R}_{k'_1/k}(G'_1)\to G/\,\mathrm{R}_{k'_1/k}(G'_1)\to G''\to 1,$$

so by induction on the number of k'_i's it suffices to treat the case when k' is a field.

Let $H=\mathrm{R}_{k'/k}(G')$, so $H(k)=G'(k')$ and $H(\mathbf{A}_k)=G'(\mathbf{A}_{k'})$ as topological groups. Since G' is an absolutely pseudo-simple k'-group that is either simply connected semisimple or basic exotic, the sequences of k-points and $\mathbf{A}_k$-points arising from (7.3.1) are exact due to Lemma 4.1.6 and Theorem 5.1.1(i). The image of T in G'' is a maximal k-split torus $T''\subseteq G''$ since $G\twoheadrightarrow G''$ has smooth kernel, and the map $T(\mathbf{A}_k)/T(\mathbf{A}_k)^1\to T''(\mathbf{A}_k)/T''(\mathbf{A}_k)^1$ between finite free $\mathbf{Z}$-modules has finite-index image (because T and T'' are k-split; see Lemma 4.2.4). Hence, for the open image U'' of U in $G''(\mathbf{A}_k)$ we see that $U''\cap T''(\mathbf{A}_k)$ has finite-index image in $T''(\mathbf{A}_k)/T''(\mathbf{A}_k)^1$, so $G''(k)\backslash G''(\mathbf{A}_k)/U''$ is finite by the hypothesis on G''. We can therefore choose a finite set of elements $g''_i\in G''(\mathbf{A}_k)$ such that $G''(\mathbf{A}_k)=\bigcup U''g''_iG''(k)$.

We may and do choose $g_i\in G(\mathbf{A}_k)$ lifting g''_i, so by surjectivity of the map $G(k)\to G''(k)$ we have

$$G(\mathbf{A}_k)=\bigcup_i Ug_iG(k)H(\mathbf{A}_k)=\bigcup_i Ug_iH(\mathbf{A}_k)G(k).$$

For each i and open subgroup $\widetilde{U}_i:=g_i^{-1}(U\cap H(\mathbf{A}_k))g_i$ in $H(\mathbf{A}_k)$, if $H(k)\backslash H(\mathbf{A}_k)/\widetilde{U}_i$ is finite then $H(\mathbf{A}_k)=\bigcup\widetilde{U}_ih_{ij}H(k)$ for a finite set $\{h_{ij}\}_{j\in J_i}\subseteq H(\mathbf{A}_k)$. Thus, assuming such finiteness for all i would give $G(\mathbf{A}_k) = \bigcup_{i,j} Ug_ih_{ij}G(k)$ (since $g_i\widetilde{U}_i \subseteq Ug_i$), thereby establishing the desired finiteness of $G(k)\backslash G(\mathbf{A}_k)/U$. Hence, it remains to show that $H(k)\backslash H(\mathbf{A}_k)/(g^{-1}(U \cap H(\mathbf{A}_k))g)$ is finite for all $g \in G(\mathbf{A}_k)$.

Step 4. If G' is k'-anisotropic then the coset space $H(k)\backslash H(\mathbf{A}_k) = G'(k')\backslash G'(\mathbf{A}_{k'})$ is compact (by Theorem 5.1.1(ii)), so the desired finiteness is clear in such cases. We now have to consider when G' is k'-isotropic. The canonical topological group isomorphism $H(\mathbf{A}_k) \simeq G'(\mathbf{A}_{k'})$ identifies $H(k)$ with $G'(k')$ and carries $U \cap H(\mathbf{A}_k)$ isomorphically onto an open subgroup $U' \subseteq G'(\mathbf{A}_{k'})$, so the following criterion will be very useful:

Lemma 7.3.3. *Let k' be a global function field and G' a k'-isotropic absolutely pseudo-simple k'-group that is either simply connected semisimple or basic exotic. For an open subgroup $U' \subseteq G'(\mathbf{A}_{k'})$, $G'(k')\backslash G'(\mathbf{A}_{k'})/U'$ is finite if and only if U' is non-compact, in which case this double coset space consists of a single point.*

Proof. By Theorem 2.3.8(ii), the basic exotic case reduces to the simply connected semisimple case. Hence, we may and do assume that G' is a connected semisimple k'-group that is absolutely simple and simply connected. Since G' is k'-isotropic, $G'(k')\backslash G'(\mathbf{A}_{k'})$ is non-compact by Theorem 5.1.1(ii). Thus, the double coset space $G'(k')\backslash G'(\mathbf{A}_{k'})/U'$ cannot be finite if U' is compact.

Now we assume that U' is non-compact and will show that there is a place v_0' of k' such that U' has non-compact projection into the factor $G'(k'_{v_0'})$. Grant this for a moment. Since G' is absolutely simple and simply connected over k', by a theorem of Tits (proved in [Pr2]) the only non-compact open subgroup of $G'(k'_{v_0'})$ is the entire group, so U' maps onto $G'(k'_{v_0'})$. We claim that the open subgroup U' in $G'(\mathbf{A}_{k'})$ must contain the entire factor group $G'(k'_{v_0'})$. Clearly $U_0' := U' \cap G'(k'_{v_0'})$ is an open subgroup of $G'(k'_{v_0'})$. The conjugates $u'U_0'u'^{-1}$ for $u' \in U'$ lie in U' and have trivial projection into the factors $G'(k'_{v'})$ for all $v' \neq v_0'$, so these conjugates are contained in U_0'. But each element of $G'(k'_{v_0'})$ occurs as the v_0'-factor of some element u' of U', so by varying u' we see that U_0' is an open normal subgroup of $G'(k'_{v_0'})$. There is no proper open normal subgroup of $G'(k'_{v_0'})$, so $U' = G'(k'_{v_0'})$.

(For the convenience of the reader, here is a proof that an open normal subgroup U_0' in $G'(k'_{v_0'})$ must be full. By the theorem of Tits [Pr2] mentioned above, it suffices to prove that U_0' is non-compact. We will construct a non-compact closed subset of U_0'. Using the $k'_{v_0'}$-points of an open Bruhat cell relative to a choice of maximal $k'_{v_0'}$-split torus $T' \neq 1$ in $G'_{k'_{v_0'}}$, there exists a nontrivial point $u_0' \in U_0'$ that lies in the root group of $(G'_{k'_{v_0'}}, T')$ for some $\lambda \in \Phi(G'_{k'_{v_0'}}, T')$. The orbit map $T' \to G'_{k'_{v_0'}}$ defined by $t' \mapsto t'u_0't'^{-1}$ lands in U_0' on $k'_{v_0'}$-points by normality, and it has image $\mathbf{G}_{\rm a} - \{0\}$ as a map of varieties. Let T'' be the codimension-1 subtorus $(\ker\lambda)^0_{\rm red}$, so the orbit map factors through $T'/T'' \simeq \mathrm{GL}_1$ with $(T'/T'')(k'_{v_0'}) = T'(k'_{v_0'})/T''(k'_{v_0'})$, and the induced map $T'/T'' \to \mathbf{G}_{\rm a} - \{0\}$ is identified with the nth-power endomorphism of GL_1 for some $n \neq 0$. We conclude that there is a closed $k'_{v_0'}$-subgroup $\mathbf{G}_{\rm a} \subset G'_{k'_{v_0'}}$ such that the map $\mathrm{GL}_1 \to \mathbf{G}_{\rm a}$ defined by $c \mapsto c^n$ has image on $k'_{v_0'}$-points contained in U_0'. Hence, $U_0' \cap \mathbf{G}_{\rm a}(k'_{v_0'})$ is a subset of $k'_{v_0'}$ that contains all nonzero nth powers in $k'_{v_0'}$ and so is non-compact. But it is also closed in U_0', so we are done.)

We conclude from the containment $G'(k'_{v'_0}) \subseteq U'$ that the natural surjective map

$$G'(k')\backslash G'(\mathbf{A}_{k'})/U' \twoheadrightarrow G'(k')\backslash G'(\mathbf{A}_{k'}^{v'_0})/(U' \cap G'(\mathbf{A}_{k'}^{v'_0}))$$

(arising from the quotient ring map $\mathbf{A}_{k'} \twoheadrightarrow \mathbf{A}_{k'}^{v'_0}$) is injective. The target is a singleton due to strong approximation for isotropic simply connected and absolutely simple semisimple groups [Pr1, Thm. A]. This completes the reduction to proving that U' has non-compact image in the local factor group $G'(k'_{v'})$ for some place v' of k'. We will assume to the contrary and seek a contradiction.

Choose a finite non-empty set S' of places of k' so that G' spreads out to a semisimple group scheme $\mathscr{G}'$ over $\mathscr{O}_{k',S'}$ with connected fibers. Increase S' so that U' contains an open subgroup of the form $K = \prod K_{v'}$ such that $K_{v'}$ is a compact open subgroup of $G'(k'_{v'})$ for all v' and is equal to the compact open subgroup $\mathscr{G}'(\mathscr{O}_{v'})$ for all $v' \notin S'$. For all $v' \notin S'$ it follows from Bruhat–Tits theory that the compact open subgroup $\mathscr{G}'(\mathscr{O}_{v'})$ in $G'(k'_{v'})$ is maximal. Suppose U' has compact projection into $G'(k'_{v'})$ for *all* places v' of k'. Since $G'(\mathbf{A}_{k'}) = G'(k'_{S'}) \times G'(\mathbf{A}_{k'}^{S'})$ topologically, the open subgroup image W' of U' in $G'(\mathbf{A}_{k'}^{S'})$ must be non-compact (as U' is non-compact). But W' contains the compact open subgroup $\prod_{v' \notin S'} K_{v'}$ whose local factors are maximal compact open subgroups, so by compactness of the local projections of W' we get $W' = \prod_{v' \notin S'} K_{v'}$, contrary to the non-compactness of W'. ■

By Lemma 7.3.3, our task in the pseudo-reductive case is reduced to showing that $U \cap H(\mathbf{A}_k)$ is non-compact when G' is k'-isotropic. Note that H is necessarily k-isotropic. Suppose that $U \cap H(\mathbf{A}_k)$ is compact. The intersection $T \cap H$ clearly contains a (unique) maximal k-split torus $T_0 \subseteq H$, and $T_0 \neq 1$ due to the maximality of T as a k-split torus in G. Since $U \cap T_0(\mathbf{A}_k)$ is compact (due to our hypothesis that $U \cap H(\mathbf{A}_k)$ is compact), it lies in $T(\mathbf{A}_k)^1$. Hence, the finite-index image of $U \cap T(\mathbf{A}_k)$ in the $\mathbf{Z}$-lattice $T(\mathbf{A}_k)/T(\mathbf{A}_k)^1$ is a $\mathbf{Z}$-lattice with rank equal to $\dim T$ and it is also a discrete torsion-free quotient of the group $(U \cap T(\mathbf{A}_k))/(U \cap T_0(\mathbf{A}_k))$. But this latter group is an open subgroup of $T(\mathbf{A}_k)/T_0(\mathbf{A}_k) = (T/T_0)(\mathbf{A}_k)$, so it has a maximal compact open subgroup modulo which it is a $\mathbf{Z}$-lattice with rank at most $\dim(T/T_0)$. Since $\dim(T/T_0) < \dim T$, we have a contradiction. This shows that $U \cap H(\mathbf{A}_k)$ is indeed non-compact, so the case of pseudo-reductive G (and hence the general case) is settled. ■

Remark 7.3.4. The connectedness hypothesis on G in Theorem 7.3.1 cannot be removed. Letting $E := G/G^0$, the problem is that even if $G^0 = T$ is a split torus, the $E(\mathbf{A}_k)$-action on $G^0(\mathbf{A}_k)$ may not preserve $T(\mathbf{A}_k)^1$ and so may interact badly with the hypothesis on U. Here is a counterexample using $G = T \rtimes \Gamma$, where $T = \mathrm{GL}_1$ and $\Gamma = \langle -1 \rangle$ with the nontrivial element of Γ acting on T via inversion.

Pick a pair of distinct places v_0 and v_1 of k, and let $a \in \mathbf{A}_k^\times$ be an idele such that the components a_{v_0} and a_{v_1} have the same *nontrivial* norm (in $q^{\mathbf{Z}}$) and $a_v = 1$ for all other v. Inside of $G(\mathbf{A}_k) = \mathbf{A}_k^\times \rtimes \prod_v \Gamma$, let

$$U = \langle a, \prod_v \mathscr{O}_v^\times \rangle \rtimes \prod_{v \neq v_0, v_1} \Gamma;$$

this makes sense as a subgroup because we drop the v_0-factor and v_1-factor from the product on the right side. (We could have instead set the components in those two factors to be equal, rather than trivial.) Obviously U is open in $G(\mathbf{A}_k)$, and $U \cap \mathbf{A}_k^\times$ has nontrivial image under the idelic norm, so $U \cap T(\mathbf{A}_k)$ has finite-index image in $T(\mathbf{A}_k)/T(\mathbf{A}_k)^1 = q^{\mathbf{Z}}$.

Consider the element $g \in \prod_v \Gamma \subset G(\mathbf{A}_k)$ with trivial components away from v_1 and nontrivial component at v_1. The conjugate $gag^{-1} \in \mathbf{A}_k^\times$ is the idele whose components away from v_0 and v_1 are trivial and components at v_0 and v_1 are respectively a_{v_0} and $a_{v_1}^{-1}$. Hence, $gag^{-1} \in T(\mathbf{A}_k)^1$! We claim that for $t \in T(\mathbf{A}_k) = \mathbf{A}_k^\times$, $G(k)tgU$ determines the idelic norm of t, so then varying over the infinitely many idelic norms gives infinitely many classes in $G(k)\backslash G(\mathbf{A}_k)/U$.

For $t' \in \mathbf{A}_k^\times$, suppose $t'g = \gamma tgu$ with $\gamma \in G(k) = k^\times \rtimes \Gamma$ and $u \in U$, so $\gamma t(gug^{-1}) = t' \in \mathbf{A}_k^\times$. Either $\gamma = c \in k^\times$ or $\gamma = (c, \Delta(-1))$, where $\Delta : \Gamma \to \prod_v \Gamma$ is the diagonal. Likewise, writing $u = (u_T, u_\Gamma)$ with $u_T \in T(\mathbf{A}_k) = \mathbf{A}_k^\times$ and $u_\Gamma = \Gamma(\mathbf{A}_k) = \prod_v \Gamma$, we have $gug^{-1} = (g.u_T, u_\Gamma)$. If $\gamma = c \in k^\times$ then $\gamma t(gug^{-1}) = (ct \cdot g.u_T, u_\Gamma)$, and if $\gamma = (c, \Delta(-1))$ then $\gamma t(gug^{-1}) = (ct \cdot g.u_T^{-1}, -u_\Gamma)$. In both cases $u_T = a^n x$ with $n \in \mathbf{Z}$ and $x \in \prod_v \mathscr{O}_v^\times \subset T(\mathbf{A}_k)^1$ by definition of U. Thus, $g.u_T = (g.a)^n(g.x) \in T(\mathbf{A}_k)^1$, so t and t' have the same idelic norm.

Corollary 7.3.5. *Let $\pi : G' \twoheadrightarrow G$ be a smooth surjective homomorphism between smooth connected groups over a global function field k, and assume that G affine. If $\ker \pi$ is connected then $G(k)\backslash G(\mathbf{A}_k)/\pi(G'(\mathbf{A}_k))$ is finite.*

Proof. Since π is smooth with connected kernel, $\pi(G'(\mathbf{A}_k))$ is an open subgroup of $G(\mathbf{A}_k)$. Let $T \subseteq G$ be a maximal k-split k-torus. We apply Proposition 3.1.3 to get a maximal k-split k-torus $T' \subseteq G'$ mapping onto T. By Theorem 7.3.1, we just have to note that the map of $\mathbf{Z}$-lattices $T'(\mathbf{A}_k)/T'(\mathbf{A}_k)^1 \to T(\mathbf{A}_k)/T(\mathbf{A}_k)^1$ has image with finite index, by Lemma 4.2.4. ■

The case of non-affine G' (with affine G) in Corollary 7.3.5 will be used at the end of the proof of Theorem 7.5.3(ii). The connectedness hypothesis on $\ker \pi$ cannot be dropped, as we see by taking $G = G' = \mathrm{GL}_1$ and π to be the nth-power map for $n > 1$ not divisible by char(k).

Corollary 7.3.6. *Let $1 \to G' \to G \to G'' \to 1$ be a short exact sequence of smooth connected affine groups over a global function field k. Assume the open image of $G(\mathbf{A}_k)$ in $G''(\mathbf{A}_k)$ is normal. The Tamagawa number τ_G is finite if and only if the Tamagawa numbers $\tau_{G'}$ and $\tau_{G''}$ are finite.*

Proof. In [Oes, III, 5.3] such an equivalence is proved conditional on two finiteness hypotheses that we now know *always* hold: the first is a special case of the conclusion of Corollary 7.3.5, and the second is an immediate consequence of Theorem 1.3.3(i). ■

The normality hypothesis in Corollary 7.3.6 is satisfied whenever G' is central in G, such as in the quotient procedure that defines the generalized standard construction of pseudo-reductive groups in Definition 2.3.3.

As another application of Theorem 7.3.1, there are analogues of the results in §7.2 for $S = \emptyset$ using the same proofs, provided that we assume G is k-anisotropic. We will not use these analogues later, so we content ourselves with stating the analogue of Proposition 7.2.1; the interested reader can formulate an analogue of Theorem 7.2.2.

Proposition 7.3.7. *Let k be a global function field and let X be the associated smooth proper geometrically connected curve over the finite field of constants $\mathbf{F}$ of k. Let $\mathscr{G}$ be an X-group scheme of finite type with affine structural morphism to X and smooth connected generic fiber $\mathscr{G}_\eta$ over $\mathbf{F}(\eta) = k$ that is k-anisotropic. The cohomology set $\mathrm{H}^1(X, \mathscr{G})$ is finite.*

In this result it is natural to try to relax the restrictive k-anisotropicity hypothesis on $\mathscr{G}_\eta$. This requires interpreting "finiteness" of cohomology in a manner other than the set-theoretic one, as the following standard examples show. Since $\mathrm{Br}(X) = 1$ by global class field theory for k, for all $n \geqslant 1$ the pointed set $\mathrm{H}^1(X, \mathrm{PGL}_n)$ is identified with the quotient $\mathrm{Vec}_n(X)/\mathrm{Pic}(X)$

of the set $\mathrm{Vec}_n(X)$ of isomorphism classes of rank-n vector bundles on X modulo twisting by line bundles. Also, $\mathrm{H}^1(X, \mathrm{SL}_n)$ maps onto the pointed set $\mathrm{Vec}_n(X)^{\det=1}$ of rank-n vector bundles with determinant 1 for any $n \geqslant 1$. Both sets $\mathrm{Vec}_n(X)/\mathrm{Pic}(X)$ and $\mathrm{Vec}_n(X)^{\det=1}$ are infinite when $n > 1$, as we see by using direct sums of line bundles on X (with varying degrees) since such a direct sum determines the unordered n-tuple of line bundles up to isomorphism. Hence, $\mathrm{H}^1(X, \mathrm{PGL}_n)$ and $\mathrm{H}^1(X, \mathrm{SL}_n)$ are infinite.

For any X-affine *flat* X-group scheme $\mathscr{G}$ of finite type with connected generic fiber, a more natural way to define the "size" of $\mathrm{H}^1(X, \mathscr{G})$ is to assign non-trivial mass to cohomology classes as follows. The fibered category $\mathrm{Bun}_{\mathscr{G}}$ of $\mathscr{G}$-torsors on X (fibered over the category of $\mathbf{F}$-schemes) is a (quasi-separated) Artin stack locally of finite type over $\mathbf{F}$ [Bro, Thm. 2.1]. The cohomology set $\mathrm{H}^1(X, \mathscr{G})$ is the set of isomorphism classes in $\mathrm{Bun}_{\mathscr{G}}(\mathbf{F})$, so this inspires assigning each $\xi \in \mathrm{H}^1(X, \mathscr{G})$ the mass $|\mathrm{Aut}_X(\xi)|^{-1}$ (this automorphism group is finite since $\mathbf{F}$ is finite) and asking if the sum of the masses (over the countably many ξ) is finite.

When the generic fiber $\mathscr{G}_\eta$ is smooth and connected with no nontrivial k-rational characters (e.g., $\mathscr{G}$ is perfect) then this sum of masses is related to the Tamagawa number $\tau_{\mathscr{G}_\eta}$ whose finiteness in general is established in §7.4. We will address the precise relationship between this refined counting procedure and Tamagawa numbers elsewhere.

7.4 Finiteness for Tamagawa numbers

In this section we prove Theorem 1.3.6. The proof will rest on several ingredients: Oesterlé's work on Tamagawa numbers in [Oes], the structure theory of pseudo-reductive groups, and the result in Corollary 7.3.5 that is a version of finiteness of class numbers in the case $S = \emptyset$.

Let G be a smooth connected affine group over a global field k. The definitions of the Tamagawa measure μ_G on $G(\mathbf{A}_k)$, the closed subgroup $G(\mathbf{A}_k)^1 \subseteq G(\mathbf{A}_k)$, and the induced measure μ_G^1 on $G(k)\backslash G(\mathbf{A}_k)^1$ (as in the discussion preceding Theorem 1.3.6) all rest on the k-group G and not merely the topological group $G(\mathbf{A}_k)$ equipped with its discrete subgroup $G(k)$. Thus, in general the definition of the Tamagawa number τ_G has the same dependence. Consequently, to prove the finiteness of τ_G for general G over a global function field, after passing to the pseudo-reductive case below we will not be able to immediately apply Theorem 2.3.8(ii) to pass to the generalized standard pseudo-reductive case when $\mathrm{char}(k) = 2$ (as we have done in all preceding considerations).

Now assume k is a global function field. Consider the short exact sequence

$$1 \to G' \xrightarrow{j} G \xrightarrow{\pi} G'' \to 1$$

with $G' = \mathscr{R}_{u,k}(G)$, so G' is solvable and G'' is pseudo-reductive. By [Oes, IV, 1.3], for any solvable smooth connected affine k-group H, the coset space $H(k)\backslash H(\mathbf{A}_k)^1$ is compact and hence has finite volume under μ_H^1. Thus, $\tau_{G'}$ is finite, so by [Oes, III, 5.2] the finiteness of τ_G is equivalent to the simultaneous finiteness of the $\mu_{G''}^1$-volume of

$$G''(k)\backslash(G''(\mathbf{A}_k)^1 \cap \pi(G(\mathbf{A}_k)))$$

(which obviously holds if $\tau_{G''}$ is finite) and of

$$\ker(Ш^1_\emptyset(k, G') \to Ш^1_\emptyset(k, G)).$$

Since $Ш^1_\emptyset(k, G')$ is finite (by the finiteness of Tate–Shafarevich sets in the solvable case [Oes, IV, 2.6(a)]), it follows that the finiteness of τ_G is reduced to that of $\tau_{G''}$. Hence, we have reduced to the case when G is pseudo-reductive. The finiteness of Tamagawa numbers in the commutative (even solvable) case has already been noted, so we may assume G is non-commutative.

If $G = G_1 \times G_2$ then it is obvious that finiteness of τ_G is equivalent to that of τ_{G_1} and τ_{G_2}. Moreover, by [Oes, II, 1.3] the measures μ_G and μ_G^1 as well as the Tamagawa number (including finiteness) are "invariant" under Weil restriction through any finite extension of global fields. Thus, by applying Theorem 2.3.6 if G_{k_s} has a non-reduced root system, and then applying Theorem 2.3.8(i), for pseudo-reductive G it suffices to separately treat two cases: G is generalized standard (in any characteristic) or $\mathrm{char}(k) = 2$ and G is a basic non-reduced pseudo-simple k-group in the sense of Definition 2.3.5.

Consider the basic non-reduced pseudo-simple case in characteristic 2. In such cases $G(\mathbf{A}_k)^1 = G(\mathbf{A}_k)$ and this is unimodular. Hence, finiteness of τ_G amounts to the condition that the coset space $G(k)\backslash G(\mathbf{A}_k)$ modulo the discrete subgroup $G(k)$ has finite volume when equipped with the induced measure from the Haar μ_G measure on $G(\mathbf{A}_k)$. Clearly the choice of Haar measure does not matter, so this problem is intrinsic to the *topological* group $G(\mathbf{A}_k)$ equipped with its discrete subgroup $G(k)$. By Theorem 2.3.8(ii), there is a surjective k-homomorphism $G \to \overline{G} \simeq \mathrm{R}_{k^{1/2}/k}(\mathrm{Sp}_{2n})$ that induces a homeomorphism $G(\mathbf{A}_k) \simeq \overline{G}(\mathbf{A}_k)$ carrying $G(k)$ onto $\overline{G}(k)$. This allows us to replace G with $\overline{G} = \mathrm{R}_{k^{1/2}/k}(\mathrm{Sp}_{2n})$.

Thus, we now may assume G is non-commutative and generalized standard with k of any nonzero characteristic. Let $(G', k'/k, T', C)$ be the generalized standard presentation adapted to a choice of maximal k-torus T (Remark 2.3.4). This yields a central extension

$$1 \to \mathrm{R}_{k'/k}(C') \xrightarrow{j} \mathrm{R}_{k'/k}(G') \rtimes C \xrightarrow{\pi} G \to 1 \tag{7.4.1}$$

where $C' = Z_{G'}(T')$ is the (commutative) Cartan k'-subgroup of G' associated to the maximal k'-torus $T' \subset G'$ corresponding to the choice of T. By Corollary 7.3.6, to prove the finiteness of τ_G it suffices to prove the finiteness of the Tamagawa number of the middle term in (7.4.1). (This application of Corollary 7.3.6 only requires finiteness of Tate-Shafarevich sets in the commutative case, so the main content is the input from Corollary 7.3.5.)

To prove that the semi-direct product term in the middle of (7.4.1) has finite Tamagawa number, by (an easy instance of) Corollary 7.3.6 it suffices to prove finiteness of the Tamagawa numbers for the factors of the semi-direct product. Let $\{k'_i\}$ be the set of factor fields of k', and G'_i the k'_i-fiber of G'. Since τ_C is finite by the commutativity of C, and

$$\tau_{\mathrm{R}_{k'/k}(G')} = \prod \tau_{\mathrm{R}_{k'_i/k}(G'_i)} = \prod \tau_{G'_i}$$

(by the invariance of Tamagawa numbers with respect to Weil restriction through finite extensions of global fields [Oes, II, 1.3]), for the middle term in (7.4.1) it remains to treat the finiteness problem for τ_G when G is either absolutely simple and simply connected or is basic exotic with $\mathrm{char}(k) \in \{2, 3\}$. The case of connected semisimple groups is well-known, so we may assume that G is basic exotic. In such cases we can use Theorem 2.3.8(ii) exactly as in the basic non-reduced pseudo-simple case above to reduce the finiteness problem to the known connected semisimple case. This takes care of the finiteness for the middle term in (7.4.1) and so completes the proof of Theorem 1.3.6.

The following corollary affirmatively answers a question raised by M. Emerton.

Corollary 7.4.1. *Let G be a smooth connected affine group over a global field k, and R its maximal k-split solvable smooth connected normal k-subgroup. The group $G(\mathbf{A}_k)/R(\mathbf{A}_k) = (G/R)(\mathbf{A}_k)$ is unimodular and the quotient $G(k)\backslash G(\mathbf{A}_k)/R(\mathbf{A}_k) = (G/R)(k)\backslash(G/R)(\mathbf{A}_k)$ by a discrete subgroup has finite volume.*

Proof. The k-split property implies $G(\mathbf{A}_k)/R(\mathbf{A}_k) = (G/R)(\mathbf{A}_k)$ and $G(k)/R(k) = (G/R)(k)$, so we may replace G with G/R to reduce to the case $R = 1$. In particular, the maximal k-split smooth connected unipotent normal k-subgroup of G is trivial, so $U := \mathscr{R}_{u,k}(G)$ is k-wound in the sense of Definition 7.1.1ff. (See [CGP, Cor. B.3.5].) Our aim is to prove that $G(\mathbf{A}_k)$ is unimodular and that $G(k)\backslash G(\mathbf{A}_k)$ has finite volume. In fact, we shall prove that $\mathrm{X}_k(G) = 1$, so $G(\mathbf{A}_k) = G(\mathbf{A}_k)^1$. This would imply the unimodularity, and the volume-finiteness is then the established finiteness of the Tamagawa number τ_G.

To prove that G has no nontrivial k-rational characters, it is equivalent to prove the same for the pseudo-reductive quotient $G' = G/U$. The key step is to show that the maximal central k-split torus Z' in G' is trivial. The preimage H of Z' in G is a solvable smooth connected affine k-group in which a maximal k-torus T maps isomorphically onto Z', so H is identified with a semidirect product $U \rtimes Z'$. But U is k-wound, so the only action on it over k by a k-torus is the trivial one [CGP, Cor. B.4.4], and hence $H = U \times Z'$. It follows that Z' is normal in G (since H is), so $Z' \subseteq R = 1$.

Now it remains to prove that if G is a pseudo-reductive k-group with no nontrivial central k-split torus then $\mathrm{X}_k(G) = 1$. This is a general fact over any field. Indeed, by [CGP, Lemma 1.2.5(iii)] any maximal k-torus T in G is an almost direct product of the maximal k-torus $T \cap \mathscr{D}(G)$ in $\mathscr{D}(G)$ and the maximal central k-torus S in G, and any k-rational character χ of G kills $\mathscr{D}(G)$ as well as the maximal k-anisotropic k-torus in S. Since S is k-anisotropic in our case, it follows that $\chi(T) = 1$. But maximal tori are carried onto maximal tori under any surjective homomorphism between smooth linear algebraic groups, so $\chi = 1$. ■

Remark 7.4.2. By [Bo1, 15.4(i)], $U := \mathscr{R}_{u,k}(R)$ is k-split and R/U is a k-split torus. In particular, U is the maximal k-split smooth connected unipotent normal k-subgroup of G, so by [CGP, Cor. B.3.5] and the discussion immediately preceding Proposition 4.1.9, R is the maximal central k-split torus in G precisely when G is quasi-reductive in the sense of Definition 4.1.8 (which includes pseudo-reductive G).

7.5 Non-affine groups

Let k be a field and X a proper algebraic space over k. By [Ar, Thm. 6.1] (and standard flatness and graph arguments with Hilbert functors), the automorphism functor $S \rightsquigarrow \mathrm{Aut}_S(X_S)$ is a (quasi-separated) algebraic space group locally of finite type over k. Hence, by [Ar, Lemma 4.2] it is represented by a k-group scheme $\mathrm{Aut}_{X/k}$ locally of finite type. Thus, the identity component $\mathrm{Aut}^0_{X/k}$ is a k-group scheme of finite type [SGA3, $\mathrm{VI_A}$, 2.4] (generally not reduced if $\mathrm{char}(k) > 0$) and the component group $\mathrm{Aut}_{X/k}/\mathrm{Aut}^0_{X/k}$ is an étale k-group whose geometric fiber can fail to be finite (e.g., $X = E \times E$ for an elliptic curve E). It is not known if this component group is always finitely generated, and in the projective case it is equivalent to ask that the image of $\mathrm{Aut}_{X/k}(\overline{k})$ in the automorphism group of the finitely generated Néron-Severi group $(\mathrm{Pic}_{X/k}/\mathrm{Pic}^0_{X/k})(\overline{k})$ is finitely generated.

One reason for interest in the structure of the automorphism group scheme is that the set of k-isomorphism classes of forms of X for the *fppf*-topology is identified with $\mathrm{H}^1(k, \mathrm{Aut}_{X/k})$. This rests crucially on the fact that we work with k-forms of X that may be algebraic spaces, even if X is a scheme. More specifically, if X is a proper k-scheme then the *fppf* k-forms of X classified by $\mathrm{H}^1(k, \mathrm{Aut}_{X/k})$ may not be schemes, due to problems with effectivity of descent (this already arises for étale k-forms of smooth 3-dimensional complete non-projective k-schemes), but if X is projective then such forms are again (projective) k-schemes.

Now assume that k is a global field. In this case it is natural to consider whether or not the pointed set

$$\mathscr{S}(X/k) := \{\text{isom. classes of } k\text{-proper algebraic spaces } X' \mid X'_{k_v} \simeq X_{k_v} \text{ for all places } v \text{ of } k\}$$

is finite. For number fields this problem was studied by Mazur in [Ma, §17–§18]. By descent theory, $\mathscr{S}(X/k)$ is naturally isomorphic to the pointed set $Ш(\mathrm{Aut}_{X/k})$, where

$$Ш(G) := Ш^1_\emptyset(k, G) = \ker(\mathrm{H}^1(k, G) \to \prod_v \mathrm{H}^1(k_v, G))$$

for any locally finite type k-group scheme G. For such G, when is $Ш(G)$ finite? For connected affine G, finiteness holds by Theorem 1.3.3(i). For abelian varieties such finiteness is the Tate–Shafarevich conjecture over k. In the Tate–Shafarevich conjecture it is essential to require local triviality at *all* places:

Example 7.5.1. Let A be an abelian variety of dimension $g > 0$ over a number field k, with $r := \mathrm{rank}_{\mathbf{Z}}(A(k))$. Fix a prime p and let S be a set of p-adic places of k. We claim $Ш^1_S(k, A)$ is infinite if $r < g\sum_{v\in S}[k_v : \mathbf{Q}_p]$. For each $n \geqslant 1$ let $M_n = A[p^n]$ and let M_n^* denote its Cartier dual. Let $\mathscr{L} = \{\mathscr{L}_v\}$ be the set of local conditions on $\mathrm{H}^1(k, M_n)$ given by the Selmer condition away from S and no local condition at places in S; that is, $\mathscr{L}_v = \mathrm{H}^1(k_v, M_n)$ for $v \in S$ and $\mathscr{L}_v = A(k_v)/(p^n)$ for $v \notin S$ (including $v|\infty$). Letting $\mathscr{L}^\perp$ denote the dual set of local conditions on $\mathrm{H}^1(k, M_n^*)$, the Wiles product formula [NSW, VIII, Thm. 8.6.20] gives

$$\frac{h^1_{\mathscr{L}}(k, M_n)}{h^1_{\mathscr{L}^\perp}(k, M_n^*)} = \frac{h^0(k, M_n)}{h^0(k, M_n^*)} \cdot \prod_v \frac{\#\mathscr{L}_v}{h^0(k_v, M_n)} = \frac{\#A(k)[p^n]}{\#A[p^n]^*(k)} \cdot \prod_v \frac{\#\mathscr{L}_v}{h^0(k_v, M_n)}, \tag{7.5.1}$$

where h^i denotes the cardinality of H^i and we form the product over all places. (Wiles' version in [Wi, Prop. 1.6] for odd-order Galois modules M has local factors differing from (7.5.1) at archimedean places and at finite places dividing $\#M$, but the product of the local discrepancies is 1 due to the global product formula for $\#M \in k^\times$.)

If $v \notin S$ then the local term at v in (7.5.1) is the Herbrand quotient for multiplication by p^n on $A(k_v)$. This is invariant under replacing $A(k_v)$ with an finite-index subgroup, so for $v|\infty$ it is $p^{-g[k_v:\mathbf{R}]n}$, for non-archimedean $v \nmid p$ it is 1, and for $v|p$ with $v \notin S$ it is $p^{g[k_v:\mathbf{Q}_p]n}$. But for $v \in S$ the local factor is

$$\frac{h^1(k_v, M_n)}{h^0(k_v, M_n)} = \frac{h^2(k_v, M_n)}{\|\#M_n\|_v} = \frac{h^0(k_v, M_n^*)}{\|\#M_n\|_v} = h^0(k_v, M_n^*)p^{2g[k_v:\mathbf{Q}_p]n}.$$

Let $A^\vee$ be the dual abelian variety. Since $A[p^n]^* \simeq A^\vee[p^n]$ via the Weil pairing, and for large n both $\#A(k)[p^n]$ and $\#A^\vee(k)[p^n]$ become constant (and likewise for k_v-points for $v \in S$), for large n we get

$$\frac{h^1_{\mathscr{L}}(k, M_n)}{h^1_{\mathscr{L}^\perp}(k, M_n)} = C \cdot \prod_{v|\infty} p^{-g[k_v:\mathbf{R}]n} \cdot \prod_{v|p, v\notin S} p^{g[k_v:\mathbf{Q}_p]n} \cdot \prod_{v\in S} p^{2g[k_v:\mathbf{Q}_p]n} = C \cdot p^{g\sum_{v\in S}[k_v:\mathbf{Q}_p]n}$$

for some $C > 0$. There is also an exact sequence

$$0 \to A(k)/(p^n) \to \mathrm{H}^1_{\mathscr{L}}(k, A[p^n]) \to Ш^1_S(k, A)[p^n] \to 0,$$

and $\#(A(k)/(p^n))$ is a constant multiple of p^{rn} for large n. But $h^1_{\mathscr{L}^\perp}(k, M_n) \geqslant 1$, so

$$\#Ш^1_S(k, A)[p^n] \geqslant C' p^{(g\sum_{v\in S}[k_v:\mathbf{Q}_p]-r)n}$$

for large n with some $C' > 0$. Thus, if $r < g\sum_{v\in S}[k_v : \mathbf{Q}_p]$ then $Ш^1_S(k, A)$ is infinite.

If one grants the Tate–Shafarevich conjecture over k then when $\mathrm{char}(k) = 0$ the finiteness of $Ш(G)$ was proved by Mazur in [Ma, §17] whenever $\Gamma := G(k_s)/G^0(k_s) = (G/G^0)(k_s)$ is finitely presented and $\Gamma \rtimes \mathrm{Gal}(K/k)$ has finitely many conjugacy classes of finite subgroups, where $K \subseteq k_s$ is the finite Galois splitting field over k for Γ. (If Γ is the constant group over k associated to an arithmetic group then these two finiteness hypotheses on Γ are satisfied. For the example $G = \mathrm{Aut}_{X/k}$ with a geometrically reduced and geometrically connected proper k-scheme X, arithmetic groups naturally intervene because Γ acts on the finitely generated Néron–Severi group of $X_{\overline{k}}$. However, verifying arithmeticity or any finiteness conditions on Γ is a difficult problem in general, and Borcherds [Bor, Ex. 5.8] has given examples of K3 surfaces X over $\mathbf{C}$ for which the image of Γ in $\mathrm{Aut}(\mathrm{NS}(X)_{\mathbf{Q}})$ is not an arithmetic group.)

Mazur's finiteness result for $Ш(G)$ over number fields (conditional on the Tate–Shafarevich conjecture over number fields and some finiteness hypotheses on G/G^0) uses characteristic 0 in an essential way. His method rests on Theorem 3.1.4, which is only available over perfect fields. To prove the function field analogue of Mazur's result we shall change the argument so that it uses Theorem 3.1.5 as a substitute for Theorem 3.1.4. In fact, our modified method also works over number fields, where it gives a simplified version of Mazur's argument (avoiding cohomological considerations over rings of S-integers). First we handle the smooth connected case, and then we address the problems introduced by the component group.

Lemma 7.5.2. *Let k be a global field. Assume the Tate–Shafarevich conjecture over k. For every smooth connected k-group G the localization map $\theta_G : \mathrm{H}^1(k, G) \to \prod \mathrm{H}^1(k_v, G)$ has finite fibers; in particular, $Ш(G)$ is finite for such G.*

Proof. By the twisting method (as reviewed at the start of §6.1), finiteness of $Ш(G) = \ker \theta_G$ for general such G implies the finiteness of fibers of the localization map. Thus, we focus on proving finiteness of $Ш(G)$.

As a first step, we treat the case when G is a semiabelian variety. We shall use a simple variant on the method in §6.3, the main issue being to make the argument work with empty S. Since G is commutative, things will simplify considerably. Consider the unique exact sequence

$$0 \to T \xrightarrow{j} G \xrightarrow{\pi} A \to 0$$

over k with T a k-torus and A an abelian variety. The formation of this sequence is compatible with passage to k-forms of G for the étale topology, so since $Ш(\mathscr{A})$ is assumed to be finite for every abelian variety $\mathscr{A}$ over k we may use the twisting method to reduce the general finiteness problem for $Ш(G)$ to the finiteness of $\ker(Ш(G) \to Ш(A))$. This kernel is $j(\theta_T^{-1}(\delta(A(\mathbf{A}_k))))$, where $\theta_T : \mathrm{H}^1(k, T) \to \prod \mathrm{H}^1(k_v, T)$ is the localization map and $\delta : A(\mathbf{A}_k) \to \bigoplus \mathrm{H}^1(k_v, T)$ is the "direct sum" of connecting maps. Our problem is therefore to show that $\theta_T^{-1}(\delta(A(\mathbf{A}_k)))$ is contained in finitely many $\delta(A(k))$-orbits in $\mathrm{H}^1(k, T)$. But θ_T has finite fibers since T is a torus, so it suffices to prove that $\delta(A(\mathbf{A}_k))/\delta(A(k))$ is finite. Even better, $\delta(A(\mathbf{A}_k))$ is finite: it is an image of $\pi(G(\mathbf{A}_k))\backslash A(\mathbf{A}_k)$, and $\pi(G(\mathbf{A}_k))$ is an open subgroup of the group $A(\mathbf{A}_k)$ that is compact since A is projective over k.

In general, by Theorem 3.1.5 there is an exact sequence

$$1 \to Z \xrightarrow{j} G^0 \xrightarrow{\pi} Q \to 1 \tag{7.5.2}$$

with a smooth *connected* commutative k-group Z satisfying $\mathscr{O}(Z) = k$ and a smooth connected affine k-group Q, and if $\mathrm{char}(k) > 0$ then Z is semi-abelian. Such an exact sequence is clearly unique, so its formation is compatible with passage to k-forms of G for the étale topology.

Theorem 1.3.3 and its number field analogue (due to Borel and Serre) imply finiteness of $Ш(Q)$, and the settled semi-abelian case (and evident injectivity of $Ш(Z) \to Ш(Z/\mathscr{R}_{u,k}(Z))$ in the number field case) yields the finiteness of $Ш(Z)$. Thus, since Z is commutative, finiteness of $Ш(G)$ is proved via the same argument as at the end of §6.3 (taking $S = \emptyset$ there) provided that $\pi(G(\mathbf{A}_k))\backslash Q(\mathbf{A}_k)/Q(k)$ is finite. In the number field case such finiteness is obvious because if $v|\infty$ then the open subgroup $\pi(G(k_v))$ has finite index in $Q(k_v)$ (so we can invoke finiteness of class numbers for Q with S taken to be the set of archimedean places). In the function field case we cannot appeal to such a trick, so the finiteness of class numbers for Q does not help if $\mathrm{char}(k) > 0$. To handle function fields we apply Corollary 7.3.5 to $\pi : G^0 \twoheadrightarrow Q$. ■

THEOREM 7.5.3. *Let k be a global field and assume the Tate–Shafarevich conjecture over k. Assume $E := G(k_s)/G^0(k_s)$ is finitely generated, and define the pointed set $Ш(E) := Ш^1_\emptyset(k, E)$ in the evident manner.*

(i) *Let K/k be a finite Galois splitting field for the $\mathrm{Gal}(k_s/k)$-action on E. If $E \rtimes \mathrm{Gal}(K/k)$ has finitely many conjugacy classes of finite subgroups (a condition visibly independent of the choice of K) then $Ш(E)$ is finite.*

(ii) *If $Ш(E)$ is finite and E is finitely presented then $Ш(G)$ is finite.*

The finiteness hypotheses on E are satisfied if E is an arithmetic group. One source of complications in the proof of part (ii) is that over rings of S-integers certain torsors are a-priori algebraic spaces rather than schemes (since finite étale covers are not cofinal among all étale covers of rings of S-integers). This issue seems to arise implicitly in [Ma].

Proof. To prove (i) we easily modify the proof of [Ma, §16, Lemma] (whose conclusion is false in nonzero characteristic) as follows. By the Chebotarev density theorem it follows that $Ш^1_\emptyset(K, E)$ is trivial, so $Ш(E) \subseteq \mathrm{H}^1(\mathrm{Gal}(K/k), E)$. This latter H^1 is finite by the group-theoretic finiteness hypothesis on $E \rtimes \mathrm{Gal}(K/k)$, completing the proof of (i).

Now we turn to (ii), so we assume $Ш(E)$ is finite and E is finitely presented. Let $G' \subseteq G$ be the smooth closed k-subgroup descending the Zariski closure of $G(k_s)$ in G_{k_s}, and define $\Gamma := G'(k_s)/(G')^0(k_s) = (G'/(G')^0)(k_s)$. Since $G'(k_s) = G(k_s)$, there is a $\mathrm{Gal}(k_s/k)$-equivariant short exact sequence of groups

$$1 \to G^0(k_s)/(G')^0(k_s) \to \Gamma \to E \to 1. \tag{7.5.3}$$

The group $G^0(k_s)/(G')^0(k_s)$ is *finite* because $G^0 \cap G'$ is an open and closed k-subgroup of G' (and so is a union of finitely many $(G')^0$-cosets). Any extension of a finitely presented group by a finitely presented group is finitely presented, so Γ must be finitely presented. The proof of (i) shows that $Ш(\Gamma) \subseteq \mathrm{H}^1(\mathrm{Gal}(K/k), \Gamma)$ for a finite Galois splitting field K/k of Γ, so by the twisting method and the finiteness of the left term in (7.5.3), the finiteness of $Ш(E)$ implies finiteness of $Ш(\Gamma)$ because $\mathrm{H}^1(\mathrm{Gal}(K/k), H)$ is finite for any finite group H equipped with a $\mathrm{Gal}(K/k)$-action.

By Remark 1.2.1, the elements of $\mathrm{H}^1(k, G)$ classify isomorphism classes of right G-torsor schemes over k. Hence, the proof of Lemma 6.1.1 works verbatim for G and thereby permits us (in view of the preceding arguments with Γ) to assume that G is *smooth*. Now E coincides with $(G/G^0)(k_s)$ equipped with its natural $\mathrm{Gal}(k_s/k)$-action. The key point is to use a finite presentation of E to make an integral model of the connected-étale sequence of G:

PROPOSITION 7.5.4. *The connected-étale sequence for G over k spreads out to an exact sequence*

$$1 \to G^0_S \to G_S \to E_S \to 1$$

of smooth separated group schemes over some ring of S-integers $\mathscr{O}_{k,S}$ such that G^0_S is quasi-projective with connected fibers and all connected components of E_S are finite étale over $\mathscr{O}_{k,S}$.

Proof. A proof is given in [Ma, §17, pp.27–28] using abstract bitorsor constructions. For the convenience of the reader who is unfamiliar with bitorsors, we now explain the method in more concrete terms.

Choose a finite Galois extension K/k splitting E that is large enough so that $G(K) \to E$ is surjective. Let S be a finite set of places of k containing the archimedean places so that K/k is unramified away from S, and let T be the set of places of K over S. Since G/G^0 is a Galois descent over k of the constant group E over K, we can uniquely spread out G/G^0 to a Galois descent E_S over $\mathscr{O}_{k,S}$ of the constant group E over $\mathscr{O}_{K,T}$. Since E is finitely presented, we may choose a finite subset $\{g_i\}$ in $G(K)$ whose image in E is a set of generators $\{\gamma_i\}$ that admits a finitely normally generated group of relations. By increasing S we can arrange that G^0 spreads out to a smooth quasi-projective $\mathscr{O}_{k,S}$-group G^0_S with connected fibers and that the conjugation action on G^0_K by each of the finitely many g_i's (uniquely) extends to an automorphism of $(G^0_S)_{\mathscr{O}_{K,T}}$. Thus, the subgroup of $G(K)$ generated by the g_i's acts on $(G^0_S)_{\mathscr{O}_{K,T}}$ extending its conjugation action on G^0_K.

Viewing the k-map $G \to G/G^0$ as a left G^0-torsor, we have a disjoint union decomposition

$$G_K = \coprod_{\gamma \in E} G^0_K \cdot [\gamma]$$

where $[\gamma] \in G(K)$ is a point in the fiber over $\gamma \in E$ that also lies in the subgroup generated by the g_i's. (The coset $G^0_K \cdot [\gamma]$ only depends on γ.) More specifically, write each γ as a word in the γ_i's and then define $[\gamma]$ to be the corresponding word in the g_i's.

The group law on G_K is given by pairings

$$G^0_K \cdot [\gamma] \times G^0_K \cdot [\gamma'] \to G^0_K \cdot [\gamma\gamma'] \tag{7.5.4}$$

for ordered pairs (γ, γ') in E. Explicitly, the group law is determined by

$$(g \cdot [\gamma], g' \cdot [\gamma']) \mapsto (g([\gamma]g'[\gamma]^{-1})([\gamma][\gamma'][\gamma\gamma']^{-1})) \cdot [\gamma\gamma'].$$

But $[\gamma]$ is in the subgroup $[E] \subseteq G(K)$ generated by the g_i's, so its conjugation action on G^0_K extends to an automorphism of $(G^0_S)_{\mathscr{O}_{K,T}}$. Also, the element $[\gamma][\gamma'][\gamma\gamma']^{-1} \in G^0(K)$ lies in the subgroup $[E]$.

Assume for a moment that $[\gamma][\gamma'][\gamma\gamma']^{-1} \in G^0_S(\mathscr{O}_{K,T})$ for all $\gamma, \gamma' \in E$. Hence, the description of the K-group G_K in terms of the K-group G^0_K and pairings (7.5.4) indexed by ordered pairs in E makes sense over $\mathscr{O}_{K,T}$, so we may spread out G_K to a smooth separated $\mathscr{O}_{K,T}$-group G_T containing $(G^0_S)_{\mathscr{O}_{K,T}}$ as an open and closed subgroup. The action on $G(K)$ by any element $\sigma \in \mathrm{Gal}(K/k)$ carries each $[\gamma] \in G(\mathscr{O}_{K,T}) \subseteq G(K)$ to a point $\sigma([\gamma]) \in G(K)$. If the point $[\sigma(\gamma)]\sigma([\gamma])^{-1} \in G^0(K)$ lies in $G^0_S(\mathscr{O}_{K,T})$ for all σ and γ then the finite Galois descent datum on G_K relative to K/k extends to one on G_T relative to $\mathscr{O}_{k,S} \to \mathscr{O}_{K,T}$. Hence, by effective Galois descent relative to the finite étale extension $\mathscr{O}_{k,S} \to \mathscr{O}_{K,T}$ we would then get the desired exact sequence of smooth separated $\mathscr{O}_{k,S}$-groups.

It remains to prove that if we increase S by a finite amount and increase T accordingly, then the elements $[\gamma][\gamma'][\gamma\gamma']^{-1} \in G^0(K)$ and $[\sigma(\gamma)]\sigma([\gamma])^{-1} \in G^0(K)$ lie in $G^0_S(\mathscr{O}_{K,T})$ for all $\gamma, \gamma' \in E$ and all $\sigma \in \mathrm{Gal}(K/k)$. In other words, we want the obstructions in $G^0(K)$ to $\gamma \mapsto [\gamma]$ being a group homomorphism or being $\mathrm{Gal}(K/k)$-equivariant to all lie in $G^0_S(\mathscr{O}_{K,T})$, at least after increasing S by a finite amount. Since E is finitely generated and $\mathrm{Gal}(K/k)$ is finite, if we can handle the obstruction to being a group homomorphism (i.e., if $[\gamma][\gamma'][\gamma\gamma']^{-1} \in G^0_S(\mathscr{O}_{K,T})$ for all $\gamma, \gamma' \in E$) then the obstruction to Galois-equivariance amounts to just finitely many more obstruction elements in $G^0(K)$ (as we see by induction on the length of words in the γ_i's). We could then certainly increase S further by a finite amount to finish the proof. Hence, the problem is to understand the obstruction to being a group homomorphism.

Now we use that E is finitely presented and not just finitely generated: the finite set of generators γ_i of E was chosen so that its group of relations is finitely normally generated. Thus, we can choose abstract words $r_1, \dots, r_N$ in the γ_i's that have trivial image in E and for which the collection of their conjugates against all abstract words in the γ_i's generates the group of relations among the γ_i's. The words $w_1, \dots, w_N$ in the g_i's in $G(K)$ corresponding to the r_j's lie in $G^0(K)$, and we may increase S by a finite amount so that these finitely many words lie in the subgroup $G^0_S(\mathscr{O}_{K,T}) \subseteq G^0(K)$. The word in the g_i's that computes $[\gamma][\gamma'][\gamma\gamma']^{-1}$ (for a choice of γ and γ') lies in $G^0(K)$ and is in the subgroup generated by the $[E]$-conjugates of the w_j's, so it also lies in $G^0_S(\mathscr{O}_{K,T})$ since $[E]$-conjugation on G^0_K respects the $\mathscr{O}_{K,T}$-structure. ∎

Fix an exact sequence as in Proposition 7.5.4, and let K/k be a finite Galois extension splitting the étale k-group $\mathscr{E} := G/G^0$ such that $G(K) \to \mathscr{E}(K) = E$ is surjective. Let T be the finite set of places of K over S. Since $E_S(\mathscr{O}_v) \to E_S(k_v) = \mathscr{E}(k_v)$ is surjective for all $v \notin S$, the map $G(k_v) \to \mathscr{E}(k_v)$ is surjective for $v \notin S$ because the map $G_S(\mathscr{O}_v) \to E_S(\mathscr{O}_v)$ is surjective (due to G^0_S being a smooth $\mathscr{O}_{k,S}$-group scheme with connected fibers).

By Lemma 7.5.2, θ_{G^0} has finite fibers. Since $Ш(E)$ is finite by hypothesis, the twisting method reduces finiteness of $Ш(G)$ to finiteness of the image of $\theta^{-1}_{G^0}(\coprod \delta(\mathscr{E}(k_v)))$ under $\mathrm{H}^1(k, G^0) \to \mathrm{H}^1(k, G)$ without requiring any finiteness assumption on $Ш(E)$ (though we may need to increase S, as the original G has been replaced with a k-form for the étale topology). Since $G(k_v) \to \mathscr{E}(k_v)$ is surjective for all $v \notin S$, the pointed set $\delta(\mathscr{E}(k_v))$ is trivial for $v \notin S$. Hence, it suffices to prove finiteness of $\delta(\mathscr{E}(k_v))$ for each $v \in S$. For any $v' \in T$ over v the restriction map $\mathrm{H}^1(k_v, G^0) \to \mathrm{H}^1(K_{v'}, G^0)$ kills $\delta(\mathscr{E}(k_v))$ because the map $G(K) \to E = \mathscr{E}(K) = \mathscr{E}(K_{v'})$ is surjective. We may therefore conclude the proof of Theorem 7.5.3(ii) when G^0 is affine by using Proposition 7.1.2.

Now we turn to the general case of Theorem 7.5.3(ii) with smooth G (assuming E is finitely presented and $Ш(E)$ is finite), so G^0 is not necessarily affine. Applying Theorem 3.1.5 to G^0 gives a unique exact sequence

$$1 \to Z \to G^0 \to Q \to 1$$

with affine Q and a smooth connected commutative k-group Z such that $\mathscr{O}(Z) = k$. Let $X \subseteq G^0$ denote the preimage of the k-unipotent radical $U = \mathscr{R}_{u,k}(Q)$, so X is an extension of U by Z and is normal in G. By Lemma 7.5.2, the localization map $\theta_X : \mathrm{H}^1(k, X) \to \prod \mathrm{H}^1(k_v, X)$ has finite fibers. In the exact sequence

$$1 \to X \xrightarrow{j} G \xrightarrow{\pi} \overline{G} \to 1 \tag{7.5.5}$$

we have that $\overline{G}^0 = Q/U$ is pseudo-reductive. (This will allow us to apply Proposition 4.1.9 to $\overline{G}^0$ at the end of the proof.) The formation of this exact sequence is compatible with passage to k-forms of G for the étale topology.

Note that $G \to \overline{G}$ induces an isomorphism on component groups. Thus, $\overline{G}$ has component group E such that $Ш(E)$ is finite by hypothesis, so since $\overline{G}^0$ is affine we conclude that $Ш(\overline{G})$ is finite (since we have settled all cases with an affine identity component). To deduce that $Ш(G)$ is finite we will use the twisting method, but beware that if G_c is an inner form of G for the étale topology arising from twisting against some right G-torsor c then $Ш(\overline{G}_c)$ might not be finite (in the original setup we only made an assumption of finiteness on $Ш(E)$, and perhaps $Ш(E_c)$ is not finite). If we enlarge S to contain all places ramified in a finite separable splitting field for the G-torsor c over k then we still have that $G_c(k_v)$ and $\overline{G}_c(k_v)$ map onto $E_c(k_v)$ for all $v \notin S$, so by finiteness of $Ш(\overline{G})$ we may then apply the twisting method to (7.5.5) to reduce the finiteness of $Ш(G)$ to the finiteness of $\mathrm{H}^1(j)(\theta_X^{-1}(\delta(\prod \overline{G}(k_v))))$ provided that we abandon (as we now do!) any hypothesis on $Ш(E)$ or $Ш(\overline{G})$, such as finiteness. Increase S so that the conclusion of Proposition 7.5.4 holds over $\mathscr{O}_{k,S}$, and let G_S be as in that proposition. Define X_S to be the schematic closure of X in G_S^0, so X_S is a quasi-projective and flat $\mathscr{O}_{k,S}$-group with generic fiber X. Increase S some more so that X_S is smooth with (geometrically) connected fibers over $\mathscr{O}_{k,S}$. The quotient $\overline{G}_S := G_S/X_S$ exists as a smooth algebraic space group over $\mathscr{O}_{k,S}$, so (7.5.5) spreads out to a short exact sequence of smooth algebraic space groups

$$1 \to X_S \to G_S \to \overline{G}_S \to 1$$

(here "short exact" means that $G_S \to \overline{G}_S$ is faithfully flat with functorial kernel X_S).

There is a natural right action of $\overline{G}(k)$ on $\mathrm{H}^1(k,X)$, and its orbits are precisely the fibers of $\mathrm{H}^1(j)$; see [Se2, I, §5.5]. By definition of this action, $\delta : \overline{G}(k) \to \mathrm{H}^1(k,X)$ is $\overline{G}(k)$-equivariant when using the right translation action of $\overline{G}(k)$ on itself. Our problem is to show that $\theta_X^{-1}(\delta(\prod \overline{G}(k_v)))$ is contained in finitely many $\overline{G}(k)$-orbits on $\mathrm{H}^1(k,X)$. Using notation as in (6.2.2), since θ_X lands in the subset $\coprod \mathrm{H}^1(k_v,X)$ (as the smooth k-group X is connected) we may replace $\delta(\prod \overline{G}(k_v))$ with $\delta(\coprod \overline{G}(k_v))$ or with the intermediate set $\delta(\overline{G}(\mathbf{A}_k))$. The connecting map $\overline{G}(k_v) \to \mathrm{H}^1(k_v,X)$ carries each $\overline{g} \in \overline{G}(k_v)$ to the isomorphism class of the right X-torsor $\pi^{-1}(\overline{g})$ over k_v, and this isomorphism class only depends upon $\overline{g}$ up to left multiplication by $\pi(G(k_v))$. Thus, the map $\delta : \overline{G}(\mathbf{A}_k) \to \coprod \mathrm{H}^1(k_v,X)$ factors through $\pi(G(\mathbf{A}_k))\backslash\overline{G}(\mathbf{A}_k)$ due to two properties: the calculation

$$\overline{G}(\mathbf{A}_k) = \varinjlim_{S' \supseteq S} \overline{G}_S(k_{S'} \times \prod_{v \notin S'} \mathscr{O}_v) = \varinjlim_{S' \supseteq S} (\prod_{v \in S'} \overline{G}(k_v) \times \overline{G}_S(\prod_{v \notin S'} \mathscr{O}_v))$$

(which rests on $\overline{G}_S$ being locally of finite presentation over $\mathscr{O}_{k,S}$) and surjectivity of the map

$$G_S(\prod_{v \notin S'} \mathscr{O}_v) \to \overline{G}_S(\prod_{v \notin S'} \mathscr{O}_v),$$

which rests on the vanishing of $\mathrm{H}^1(\mathrm{Spec}(\prod_{v \notin S'} \mathscr{O}_v), X_S)$. To prove this latter vanishing, first note that for $R := \prod_{v \notin S'} \mathscr{O}_v$ the cohomology set $\mathrm{H}^1(\mathrm{Spec}(R), X_S)$ classifies isomorphism classes of algebraic spaces $\mathscr{T}$ over R equipped with a structure of X_S-torsor for the étale topology over R. Hence, we have to show $\mathscr{T}(R) \neq \emptyset$ for such $\mathscr{T}$. To prove this, a key point is to first verify that every algebraic space over $\mathrm{Spec}(R)$ that is an X_S-torsor (for the étale topology) is necessarily a (quasi-projective) scheme.

Recall that X_S is quasi-projective over $\mathscr{O}_{k,S}$, so any algebraic space over R that is an X_S-torsor is the solution to an étale descent problem for a quasi-projective and finitely presented scheme over an étale cover of $\mathrm{Spec}(R)$. We have to prove the effectivity of such descent problems in the category of schemes. Although descent through an étale covering map $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ between affine schemes can fail to be effective even for a quasi-projective and finitely presented A'-scheme, when A' is A-*finite* it is always effective. Indeed, by standard limit arguments we may assume that $\mathrm{Spec}(A)$ is noetherian and connected, in which case a cofinal system of finite étale covers is given by connected Galois covers. Descent through such Galois coverings is a special case of descent relative to the free action of a finite group, and such descent is always effective for quasi-projective schemes (with the descent also quasi-projective). Hence, to prove the desired scheme property for X_S-torsor algebraic spaces over R, it suffices to prove that a cofinal system of étale covers of $\mathrm{Spec}(R)$ is given by *finite* étale covers. Such cofinality is a consequence of the following lemma.

Lemma 7.5.5. *Let $\{R_i\}$ be a (possibly infinite) collection of henselian local rings, and let $R = \prod R_i$. For any collection of local finite étale extensions $R_i \to R_i'$ with bounded degree,* $\mathrm{Spec}(\prod R_i')$ *is a finite étale cover of* $\mathrm{Spec}(R)$*. Moreover, a cofinal system of étale covers of* $\mathrm{Spec}(R)$ *is given by finite étale covers of this type.*

Proof. The essential issue in the argument is to handle the fact that the functor Spec does not carry infinite products over to disjoint unions. Let us first check that for any collection of local finite étale extensions $R_i \to R_i'$ with bounded degree, and $R' := \prod R_i'$, the map $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is a finite étale covering. The partition of I according to the values of the constant degree of R_i' over R_i is a finite partition, so we may reduce to the case when these degrees are the same for all i, say degree $d > 0$. By the henselian property of R_i, there is an R_i-algebra isomorphism $R_i' \simeq R_i[x]/(f_i)$ for a monic $f_i \in R_i[x]$ with degree d and separable irreducible reduction over the residue field of R_i. Hence, for the monic polynomial $f \in R[x]$ of degree d having ith component f_i for all i, we get $R' \simeq R[x]/(f)$ as R-algebras. Since the discriminant $\mathrm{disc}(f) \in R = \prod R_i$ is a unit, $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is a finite étale cover.

Now we show that such étale covers of $\mathrm{Spec}(R)$ are cofinal. Let k_i denote the residue field of R_i, so $\mathrm{Spec}(\prod k_i)$ is a closed subscheme of $\mathrm{Spec}(R)$. We claim that this closed subscheme contains all closed points of $\mathrm{Spec}(R)$, or equivalently any open subscheme $U \subseteq \mathrm{Spec}(R)$ that contains $\mathrm{Spec}(\prod k_i)$ is equal to $\mathrm{Spec}(R)$. Since $\mathrm{Spec}(\prod k_i)$ is quasi-compact, it suffices to prove that if a *quasi-compact* open subscheme U in $\mathrm{Spec}(R)$ contains all of the points $\mathrm{Spec}(k_i)$ then $U = \mathrm{Spec}(R)$. We may replace U with the union of a finite collection of affine open subschemes $\mathrm{Spec}(R_{r_1}), \dots, \mathrm{Spec}(R_{r_N})$. The condition that $\bigcup \mathrm{Spec}(R_{r_j})$ contains every $\mathrm{Spec}(k_i)$ is unaffected by replacing each r_j with its multiple $r_j' \in R = \prod R_i$ obtained by replacing each non-unit component with 0 and each unit component with 1. That is, we may assume that each r_j is an idempotent, and the condition of covering the subset of points $\{\mathrm{Spec}(k_i)\}_{i \in I}$ implies that for every $i \in I$ some r_j has ith component equal to 1 (rather than 0). It is therefore clear that the r_j's generate 1 in R, so the $\mathrm{Spec}(R_{r_j})$'s cover $\mathrm{Spec}(R)$, as desired. This argument also shows that a cofinal system of finite open coverings of $\mathrm{Spec}(R)$ is given by the finite disjoint open decompositions corresponding to finite partitions of I.

By [EGA, IV_4, 18.4.6(ii)], a cofinal system of étale covers of $\mathrm{Spec}(R)$ is given by finite collections of basic affine open subschemes $U_j \subseteq \mathrm{Spec}((R[x]/(f_j))_{f_j'})$ for monic $f_j \in R[x]$ such that $\mathrm{Spec}(R)$ is covered by the open images of the U_j's. The open images of such U_j's constitute a finite open cover of $\mathrm{Spec}(R)$, so by passing to a finite partition of I we reduce to considering an étale cover given by a single basic affine open U in $\mathrm{Spec}((R[x]/(f))_{f'})$ for some monic $f \in R[x]$ with $d := \deg(f) > 0$. In particular, if $f_i \in R_i[x]$ denotes the ith component of f then $U \cap \mathrm{Spec}((R_i[x]/(f_i))_{f_i'}) = U \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R_i)$ maps onto $\mathrm{Spec}(R_i)$. This says that the monic reduction $\overline{f}_i \in k_i[x]$ has a separable irreducible factor $\overline{g}_i$ such that (i) $\overline{f}_i = \overline{g}_i \overline{h}_i$ with $\gcd(\overline{g}_i, \overline{h}_i) = 1$ and (ii) the isolated point $\mathrm{Spec}(k_i[x]/(\overline{g}_i))$ in the special fiber of $\mathrm{Spec}(R_i[x]/(f_i)) \to \mathrm{Spec}(R_i)$ lies in U. Clearly $1 \leqslant \deg(\overline{g}_i) \leqslant d$ for all i, so by partitioning I according to the finitely many possible values of $\deg(\overline{g}_i)$ we may reduce to the case when $\deg(\overline{g}_i)$ is the same for all i.

Since R_i is henselian, by [EGA, IV_4, 18.5.13(a$''$)] there is a unique monic factorization $f_i =$

$g_i h_i$ such that g_i and h_i respectively lift $\overline{g}_i$ and $\overline{h}_i$, and moreover the map

$$R_i[x]/(f_i) \to (R_i[x]/(g_i)) \times (R_i[x]/(h_i))$$

is an isomorphism. The polynomial h_i has reduction $\overline{h}_i$ that is a unit in the reduction of $R_i[x]/(g_i)$, as does g_i', so localizing at $f_i' = h_i' g_i + g_i' h_i$ gives

$$\mathrm{Spec}((R_i[x]/(f_i))_{f_i'}) = \mathrm{Spec}(R_i[x]/(g_i)) \coprod \mathrm{Spec}((R_i[x]/(h_i))_{h_i' g_i})$$

since $\overline{g}_i$ is separable. Hence, for the henselian local $R_i' = R_i[x]/(g_i)$ and residue fields $k_i' = k_i[x]/(\overline{g}_i)$, the *quasi-compact* open subscheme $U \cap \mathrm{Spec}(\prod R_i')$ in $\mathrm{Spec}(\prod R_i')$ contains every $\mathrm{Spec}(k_i')$ and so equals $\mathrm{Spec}(\prod R_i')$. Thus, $\mathrm{Spec}(\prod R_i') \to \mathrm{Spec}(R)$ factors through U. ■

Continuing with the proof of the general case of Theorem 7.5.3(ii), by Lemma 7.5.5 and the argument preceding it we have shown that the algebraic space $\mathscr{T}$ is a quasi-projective R-scheme. We claim that $\mathscr{T}(R) = \prod_{v \notin S'} \mathscr{T}(\mathscr{O}_v)$. More generally, we claim that if Z is any quasi-compact separated scheme and $\{R_i\}_{i \in I}$ is any collection of local rings, the natural map of sets $h_Z : Z(\prod R_i) \to \prod Z(R_i)$ is bijective. This is clear when Z is affine, and in general injectivity follows from separatedness (since pullback of the quasi-coherent ideal of the diagonal under a map $\mathrm{Spec}(\prod R_i) \to Z \times Z$ gives an ideal in $\prod R_i$, and this ideal vanishes if and only if its projection into each R_i vanishes). For surjectivity of h_Z, let $\{U_1, \dots, U_n\}$ be a finite *affine* open covering of Z and choose $z_i \in Z(R_i)$ for all $i \in I$, so by locality of the R_i's each z_i factors through one of finitely many open affines $U_1, \dots, U_n$ that cover Z. We may choose a finite partition $\{I_1, \dots, I_n\}$ of I so that z_i factors through U_j for all $i \in I_j$, and then we get a point $\mathfrak{z}_j \in U_j(\prod_{i \in I_j} R_i)$ inducing z_i for all $i \in I_j$ because each U_j is affine. The points $\mathfrak{z}_1, \dots, \mathfrak{z}_n$ are points of Z valued in the respective rings $\prod_{i \in I_1} R_i, \dots, \prod_{i \in I_n} R_i$, so collectively they define a point $z \in Z(\prod_{i \in I} R)$ valued in the product of these n rings. It is clear that $h_Z(z) = (z_i)_{i \in I}$, as desired.

Our description of $\mathscr{T}(R)$ as the product $\prod_{v \notin S'} \mathscr{T}(\mathscr{O}_v)$ reduces the assertion $\mathscr{T}(R) \neq \emptyset$ to the assertion that $\mathscr{T}(\mathscr{O}_v)$ is non-empty for all $v \notin S'$. But $\mathrm{H}^1(\mathscr{O}_v, X_S) = 1$ for all $v \notin S$ by finiteness of the residue field at such v and the connectedness and smoothness of the fibers of X_S, so every X_S-torsor over $\mathscr{O}_v$ is trivial. In other words, $\mathscr{T}(\mathscr{O}_v) \neq \emptyset$ for each $v \notin S$, so indeed $\mathscr{T}(R) \neq \emptyset$.

By finiteness of the fibers of the $\overline{G}(k)$-equivariant θ_X, our problem is now reduced to showing that the set $\pi(G(\mathbf{A}_k)) \backslash \overline{G}(\mathbf{A}_k) / \overline{G}(k)$ is finite. It therefore suffices to prove the finiteness of

$$\pi(G(\mathbf{A}_k)) \backslash \overline{G}(\mathbf{A}_k) / \overline{G}^0(k).$$

If $v \notin S$ then $\overline{G}(k_v)$ and $G(k_v)$ both map onto $(G/G^0)(k_v)$ (by Lang's theorem and the choice of S), so $\pi(G(k_v)) \backslash \overline{G}(k_v) = \pi(G^0(k_v)) \backslash \overline{G}^0(k_v)$ for $v \notin S$. Thus, we obtain that $\pi(G(\mathbf{A}_k^S)) \backslash \overline{G}(\mathbf{A}_k^S) = \pi(G^0(\mathbf{A}_k^S)) \backslash \overline{G}^0(\mathbf{A}_k^S)$. The map $G \to \overline{G}$ induces an isomorphism between component groups, so we get an inclusion

$$\pi(G^0(\mathbf{A}_k)) \backslash \overline{G}^0(\mathbf{A}_k) \hookrightarrow \pi(G(\mathbf{A}_k)) \backslash \overline{G}(\mathbf{A}_k)$$

and hence a product decomposition

$$\pi(G(\mathbf{A}_k)) \backslash \overline{G}(\mathbf{A}_k) = (\pi(G(k_S)) \backslash \overline{G}(k_S)) \times (\pi(G^0(\mathbf{A}_k^S)) \backslash \overline{G}^0(\mathbf{A}_k^S)). \tag{7.5.6}$$

Since $\overline{G}^0$ is affine (though G^0 generally is not) and $\ker \pi$ is smooth and connected, the double coset space $\pi(G^0(\mathbf{A}_k)) \backslash \overline{G}^0(\mathbf{A}_k) / \overline{G}^0(k)$ is finite by Corollary 7.3.5. We will show below that the open subgroup $\pi(G(k_S))$ in $\overline{G}(k_S)$ has finite index, so $\pi(G(k_S))$ then contains a subgroup N that is normal (and even open) in $\overline{G}(k_S)$ with finite index. Thus, we may then choose a finite set of double-coset representatives $\{x_1, \dots, x_n\} \subseteq \overline{G}^0(\mathbf{A}_k)$ for $\pi(G^0(\mathbf{A}_k)) \backslash \overline{G}^0(\mathbf{A}_k) / \overline{G}^0(k)$, and a finite set $\{y_1, \ldots, y_m\}$ of representatives in $\overline{G}(k_S)$ for the quotient group $\overline{G}(k_S)/N$. It is then easy to see that $\overline{G}(\mathbf{A}_k)$ is covered by the double cosets $\pi(G(\mathbf{A}_k))(y_j, 1)x_i\overline{G}^0(k)$ for $1 \leqslant j \leqslant m$ and $1 \leqslant i \leqslant n$, so we would be done.

We are now reduced to checking that for each of the finitely many $v \in S$, the coset space $\pi(G(k_v))\backslash\overline{G}(k_v)$ is finite. The argument will apply to any place v of k. Since $\overline{G}^0(k_v)$ has finite index in $\overline{G}(k_v)$, it suffices to prove that $\pi(G^0(k_v))$ has finite index in $\overline{G}^0(k_v)$. The map $\pi : G^0 \to \overline{G}^0$ is a smooth surjection, so by Proposition 3.1.3 a maximal k_v-split torus in $\overline{G}^0_{k_v}$ is the image of a maximal k_v-split torus in $G^0_{k_v}$. By separability of k_v/k, the k_v-group $\overline{G}^0_{k_v}$ is *pseudo-reductive*. Hence, the criterion in Proposition 4.1.9 is applicable to the open subgroup $\pi(G^0(k_v)) \subseteq \overline{G}^0(k_v)$, so this has finite index. ■

Appendix A. A properness result

A.1 Main result

This appendix is largely devoted to proving the following theorem that is crucial in §5 and in the proof of Theorem 7.3.1, which in turn underlies the proof of finiteness of Tamagawa numbers in §7.4. (In §A.5 we give another application, not used elsewhere: the function field case of a general compactness criterion for $G(k)\backslash G(\mathbf{A}_k)^1$, with G any smooth connected affine group over a global field k.)

Theorem A.1.1. *Let G be a connected affine group scheme of finite type over a global field k and let $H \subseteq G$ be a closed normal subgroup scheme such that $(H_{\overline{k}})^0_{\mathrm{red}}$ is solvable. Let $\overline{G} = G/H$ be the connected affine quotient. The natural map of topological spaces*

$$G(k)\backslash G(\mathbf{A}_k)^1 \to \overline{G}(k)\backslash\overline{G}(\mathbf{A}_k)$$

is proper, where $G(\mathbf{A}_k)^1$ is defined as in Definition 4.2.2.

We only use Theorem A.1.1 when G, H, and $\overline{G}$ are smooth, but $\overline{G}$ is not necessarily reductive (e.g., we need to permit the pseudo-reductive case). The proof proceeds by reduction to the case of semisimple $\overline{G}$, and to carry out such a reduction step it seems essential to avoid smoothness hypotheses on H (when k is a function field). For this reason, we decided to eliminate smoothness hypotheses on G and $\overline{G}$ as well since it required no new ideas to do so.

As a first step in the proof of Theorem A.1.1, we wish to reduce to the case when G is a connected reductive group and H^0 is its maximal central torus (so $\overline{G}$ is smooth and semisimple). We initially aim to reduce to a slightly more general situation in which the unipotent radical of $(H^0_{\overline{k}})_{\mathrm{red}}$ descends to a *k-split* smooth connected unipotent k-subgroup of H. To motivate our argument, first suppose that k has characteristic 0. In this case the pullback in G of the radical of $\overline{G}$ is smooth and may be renamed as H since if a composite map of topological spaces $X \to Y \to Z$ is proper and $Y \to Z$ is separated then $X \to Y$ is proper. By the perfectness of number fields, we have therefore reduced to the case that $\overline{G}$ is semisimple and H contains a k-split smooth connected unipotent normal k-subgroup U such that H^0/U is a torus. To get to the same situation in nonzero characteristic we need to do some work. Most of §A.2 is devoted to carrying out this reduction step in nonzero characteristic.

A.2 Reduction to the reductive case

Lemma A.2.1. *Let k be a global field and let*

$$1 \to H' \to H \to H'' \to 1$$

be a short exact sequence of affine k-group schemes of finite type such that H' is smooth and connected with trivial degree-1 cohomology over k and over each k_v. For every place v the map $H(k_v) \to H''(k_v)$ is a topological fibration, and the map $H(\mathbf{A}_k) \to H''(\mathbf{A}_k)$ is a topological fibration. Also,

$$H(k)\backslash H(\mathbf{A}_k) \to H''(k)\backslash H''(\mathbf{A}_k)$$

is a topological fibration whose fibers are all homeomorphic to $H'(k)\backslash H'(\mathbf{A}_k)$.

Proof. By descent theory, $H \to H''$ is a smooth morphism since H' is smooth. The cohomological hypothesis on H' ensures that each map $H(k_v) \to H''(k_v)$ is surjective. Since smooth maps are Zariski-locally on the source expressed as étale over an affine space, by the local structure theorem for étale morphisms [EGA, IV_4, 18.4.6(ii)] and the classical theorem on continuity of simple roots of a varying monic polynomial of fixed degree over k_v it follows that the surjective continuous open map of topological groups $H(k_v) \to H''(k_v)$ admits continuous sections locally on $H''(k_v)$, with such sections existing through any point of $H(k_v)$. In particular, using the topological group structure on $H(k_v)$ shows that this map is a topological fibration.

To prove the analogous result for the adelic points, we first "spread out" the given short exact sequence of k-groups to a short exact sequence

$$1 \to \mathscr{H}' \to \mathscr{H} \to \mathscr{H}'' \to 1$$

of affine flat group schemes of finite type over $\operatorname{Spec} \mathscr{O}_{k,S}$ for a finite non-empty set S of places of k containing the archimedean places. (By "short exact sequence" we mean that $\mathscr{H} \to \mathscr{H}''$ is faithfully flat with functorial kernel $\mathscr{H}'$.) Increasing S enables us to arrange that $\mathscr{H}'$ is $\mathscr{O}_{k,S}$-smooth with geometrically connected fibers. Thus, by descent theory the map $\mathscr{H} \to \mathscr{H}''$ is a smooth morphism and expresses $\mathscr{H}$ as an $\mathscr{H}'$-torsor over $\mathscr{H}''$ for the étale topology. It is then a standard consequence of Lang's theorem that the map of topological groups $\mathscr{H}(\mathscr{O}_v) \to \mathscr{H}''(\mathscr{O}_v)$ (for $v \notin S$) is surjective for all $v \notin S$; see the end of Appendix C for a review of that deduction.

For $v \notin S$ the surjective map $\mathscr{H}(\mathscr{O}_v) \to \mathscr{H}''(\mathscr{O}_v)$ is induced by the fibration map $H(k_v) \to H''(k_v)$ via restriction to open subgroups, so we can construct local cross-sections for the map of topological groups $\mathscr{H}(\mathscr{O}_v) \to \mathscr{H}''(\mathscr{O}_v)$ for such v. Hence, this map on $\mathscr{O}_v$-points is a fibration for all $v \notin S$. But $\mathscr{H}''(\mathscr{O}_v)$ is compact and has a topological base of compact open sets for all $v \notin S$, so there exists a global cross-section to $\mathscr{H}(\mathscr{O}_v) \to \mathscr{H}''(\mathscr{O}_v)$. It then follows that the map of topological groups $H(\mathbf{A}_k) \to H''(\mathbf{A}_k)$ is surjective and admits local cross-sections (for the adelic topology) through any point of $H(\mathbf{A}_k)$, and so it is a fibration.

Finally, consider the natural map

$$\pi : H(k)\backslash H(\mathbf{A}_k) \to H''(k)\backslash H''(\mathbf{A}_k).$$

By the discreteness of $H''(k)$ in $H''(\mathbf{A}_k)$, the map

$$H''(\mathbf{A}_k) \to H''(k)\backslash H''(\mathbf{A}_k)$$

has local cross-sections, so we get local cross-sections for π via local cross-sections for $H(\mathbf{A}_k) \to H''(\mathbf{A}_k)$. Since $H(k) \to H''(k)$ and $H(\mathbf{A}_k) \to H''(\mathbf{A}_k)$ are surjective, the right action by $H'(\mathbf{A}_k)$ on $H(k)\backslash H(\mathbf{A}_k)$ is transitive on fibers of π and all fibers are homeomorphic. Thus, all fibers of π are homeomorphic to $\pi^{-1}(1)$.

The right action of $H'(\mathbf{A}_k)$ on fibers is continuous, and the stabilizer in $H'(\mathbf{A}_k)$ for the identity coset in $H(k)\backslash H(\mathbf{A}_k)$ is $H'(\mathbf{A}_k) \cap H(k) = H'(k)$. Since $\pi^{-1}(1)$ is closed, it follows from Theorem 4.2.1 that the natural map $H'(k)\backslash H'(\mathbf{A}_k) \to \pi^{-1}(1)$ is a homeomorphism. By the discreteness of $H'(k)$ in $H'(\mathbf{A}_k)$ and of $H(k)$ in $H(\mathbf{A}_k)$, we can use the local cross-sections and the $H'(\mathbf{A}_k)$-action to verify that π is a topological fibration, since the topological diagram

$$\begin{array}{ccc} H'(k)\backslash H(\mathbf{A}_k) & \longrightarrow & H''(\mathbf{A}_k) \\ \downarrow & & \downarrow \\ H(k)\backslash H(\mathbf{A}_k) & \xrightarrow[\pi]{} & H''(k)\backslash H''(\mathbf{A}_k) \end{array}$$

is cartesian. ■

To go further, we need to review Frobenius morphisms. If $Y \to Z$ is a map of $\mathbf{F}_p$-schemes (the case $Z = \operatorname{Spec} k$ for a field k will be of most interest to us), then $Y^{(p^n)}$ denotes the Z-scheme $Y \times_{Z,F_Z^n} Z$, where $F_Z : Z \to Z$ is the absolute Frobenius map (identity on topological spaces, pth-power map on the structure sheaf); loosely speaking, $Y^{(p^n)}$ is the Z-scheme obtained from Y by raising the coefficients in the defining equations of Y (over Z) to the p^nth power.

Definition A.2.2. The *n-fold relative Frobenius map* $F_{Y/Z,n} : Y \to Y^{(p^n)} = Y \times_{Z,F_Z^n} Z$ is the morphism whose components are $F_Y^n : Y \to Y$ and the structure map $Y \to Z$.

Loosely speaking, $F_{Y/Z,n}$ corresponds to the p^nth-power map in local coordinates (over Z). The formation of both $Y^{(p^n)}$ and $F_{Y/Z,n}$ commutes with any base change on Z and with fiber products over Z, and is functorial in the Z-scheme Y. In particular, if Y is a Z-group scheme then $F_{Y/Z,n}$ is a homomorphism of Z-group schemes.

Lemma A.2.3. *Let $H \to H'$ be a radiciel surjective homomorphism between affine group schemes of finite type over a global function field k. The natural map $H(k)\backslash H(\mathbf{A}_k) \to H'(k)\backslash H'(\mathbf{A}_k)$ is a closed embedding.*

Proof. Let $p = \operatorname{char}(k) > 0$. Using relative Frobenius morphisms in the sense of Definition A.2.2, if n is sufficiently large then by [SGA3, $\mathrm{VII_A}$, 8.3] the quotients $H_n = H/\ker F_{H/k,n}$ and $H'_n = H'/\ker F_{H'/k,n}$ are k-smooth. Consider the evident commutative diagram

$$\begin{array}{ccc} H & \longrightarrow & H' \\ \downarrow & & \downarrow \\ H_n & \longrightarrow & H'_n \end{array}$$

The two vertical maps are finite flat quotient maps, and the bottom side is also finite flat since the source and target are smooth and equidimensional and the map is a surjective homomorphism with finite fibers. To prove the theorem for the top arrow it obviously suffices to prove it for the other three sides. Hence, we can assume that $H \to H'$ is finite flat. In this case $H' = H/K$ for a finite infinitesimal normal closed subgroup scheme K in H. Such a K is killed by its own n-fold relative Frobenius morphism for some $n \geqslant 0$, so by the universal property of flat quotients we see that the corresponding relative Frobenius morphism $H \to H^{(p^n)}$ for H factors through the map $H \to H'$. In general, if $f : X \to Y$ and $f' : Y \to Z$ are continuous maps between topological spaces with f' separated and $f' \circ f$ a closed embedding, then f is a closed embedding. Thus, it suffices to treat a relative Frobenius morphism $F_{H/k,n} : H \to H^{(p^n)}$ (with possibly non-smooth H, so $F_{H/k,n}$ may not be flat).

Let $k' = k^{1/p^n}$, a global field of degree p^n over k. We have a natural isomorphism of topological groups $H^{(p^n)}(\mathbf{A}_k) \simeq H(\mathbf{A}_{k'})$, and this carries $H^{(p^n)}(k)$ over to $H(k')$. The composite map

$$H(\mathbf{A}_k) \stackrel{F_{H/k,n}}{\to} H^{(p^n)}(\mathbf{A}_k) \simeq H(\mathbf{A}_{k'})$$

is induced by functoriality with respect to the inclusion map of k-algebras $\mathbf{A}_k \to \mathbf{A}_{k'}$, and likewise the map

$$H(k) \stackrel{F_{H/k,n}}{\to} H^{(p^n)}(k) \simeq H(k')$$

is induced by functoriality with respect to the inclusion map of fields $k \to k'$. Hence, our problem is to prove that the natural map $H(k)\backslash H(\mathbf{A}_k) \to H(k')\backslash H(\mathbf{A}_{k'})$ is a closed embedding.

Rather more generally, for any finite extension of global function fields k'/k and any affine finite type k-group H, the natural map

$$\theta : H(k)\backslash H(\mathbf{A}_k) \to H(k')\backslash H(\mathbf{A}_{k'})$$

is a closed embedding. To see this, let $H' = \mathrm{R}_{k'/k}(H_{k'})$, so θ is topologically identified with the map

$$H(k)\backslash H(\mathbf{A}_k) \to H'(k)\backslash H'(\mathbf{A}_k)$$

induced by the canonical closed immersion $H \to H'$. By applying Lemma 4.2.5 to the map $H \to H'$, the natural map

$$H(k)\backslash H(\mathbf{A}_k)^1 \to H'(k)\backslash H'(\mathbf{A}_k)^1 \tag{A.2.1}$$

is a closed embedding. Due to the topological structure of the idelic norm in the case of global function fields, $H(k)\backslash H(\mathbf{A}_k)$ is topologically a disjoint union of copies of $H(k)\backslash H(\mathbf{A}_k)^1$. More precisely, if we let $\Lambda = \mathrm{Hom}(\mathrm{X}_k(H), q^{\mathbf{Z}})$ (where q is the size of the finite constant field of k) then each $h \in H(\mathbf{A}_k)$ induces an element of Λ via $\chi \mapsto \|\chi(h)\|_k$, so if $\Lambda_H \subseteq \Lambda$ is the subgroup of such elements obtained from elements $h \in H(\mathbf{A}_k)$ and if for each $\lambda \in \Lambda_H$ we choose $h_\lambda \in H(\mathbf{A}_k)$ giving rise to λ in this way then topologically we have

$$H(k)\backslash H(\mathbf{A}_k) = \coprod_{\lambda \in \Lambda_H} H(k)\backslash H(\mathbf{A}_k)^1 \cdot h_\lambda.$$

An analogous decomposition holds for $H'(k)\backslash H'(\mathbf{A}_k)$ using $\Lambda' := \mathrm{Hom}(\mathrm{X}_k(H'), q^{\mathbf{Z}})$ and its subgroup $\Lambda'_{H'}$ defined in a manner similar to Λ_H.

The natural map $\mathrm{X}_k(H')_{\mathbf{Q}} \to \mathrm{X}_k(H)_{\mathbf{Q}}$ is surjective because if $\chi : H \to \mathrm{GL}_1$ is a homomorphism then $\chi^{[k':k]}$ factors as

$$H \longrightarrow H' \xrightarrow{\mathrm{R}_{k'/k}(\chi_{k'})} \mathrm{R}_{k'/k}(\mathrm{GL}_1) \xrightarrow{\mathrm{N}_{k'/k}} \mathrm{GL}_1$$

due to the functoriality of Weil restriction and the fact that $\mathrm{GL}_1 \to \mathrm{R}_{k'/k}(\mathrm{GL}_1) \to \mathrm{GL}_1$ is raising to the $[k' : k]$th power. Thus, Λ_H is naturally identified with a subgroup of $\Lambda'_{H'}$. For each $\lambda \in \Lambda_H$ and associated choice $h_\lambda \in H(\mathbf{A}_k)$ we can use the image of h_λ in $H'(\mathbf{A}_k)$ as the corresponding choice $h'_{\lambda'}$ for the image λ' of λ in $\Lambda'_{H'}$. In this way, the initial map that we want to be a closed embedding is identified with a disjoint union of copies of the closed embedding (A.2.1), followed by a further open and closed embedding. ∎

Now we can reduce the proof of Theorem A.1.1 to the case when H and G are smooth (so $\overline{G} = G/H$ is also smooth), as follows. We may and do assume $\mathrm{char}(k) = p > 0$. The trick for passing to smooth groups in Lemma 3.1.1 is useful for problems involving rational points and cohomology, but is not useful for problems involving quotients (since the smoothening process via Lemma 3.1.1 is poorly behaved with respect to quotients). On the other hand, when working with quotients there is an alternative smoothening process that is rather convenient and was already used in the proof of Lemma A.2.3 (but is not so useful for problems involving rational points or cohomology): pass to the quotient by the kernel of a sufficiently high Frobenius iterate. More specifically, by [SGA3, $\mathrm{VII_A}$, 8.3], for sufficiently large n the quotient $\overline{G}_n := \overline{G}/\ker F_{\overline{G}/k,n}$ is k-smooth.

Since $\ker F_{\overline{G}/k,n}$ is an infinitesimal normal k-subgroup scheme in $\overline{G} = G/H$, its scheme-theoretic preimage H_n in G is a closed normal subgroup scheme containing H that has the same underlying topological space as H. Hence, the quotient H_n/H has no nontrivial geometric points, so it is an infinitesimal group. Moreover, the map $H(\overline{k}) \to H_n(\overline{k})$ on $\overline{k}$-points is an isomorphism of abstract groups, and the map of étale component groups $\pi_0(H) \to \pi_0(H_n)$ is an isomorphism (as it is a radiciel surjection between étale k-groups). The equality $H(\overline{k}) = H_n(\overline{k})$ implies that $((H_n)_{\overline{k}})^0_{\mathrm{red}}$ is solvable. Since Lemma A.2.3 may be applied to the map $\overline{G} \to \overline{G}_n$, we may replace H with H_n to reduce to the case that $\overline{G}$ is k-smooth.

By applying Lemma A.2.3 to the n-fold relative Frobenius morphisms $G \to G^{(p^n)}$ and $\overline{G} \to \overline{G}^{(p^n)}$ for any $n \geqslant 0$, to prove that $G(k)\backslash G(\mathbf{A}_k)^1 \to \overline{G}(k)\backslash \overline{G}(\mathbf{A}_k)$ is proper it suffices to check the analogous assertion for the map $G^{(p^n)} \to \overline{G}^{(p^n)}$ (with kernel $H^{(p^n)}$) induced via base change along the p^n-power endomorphism of k for some $n \geqslant 0$. Using the isomorphism $k^{1/p^n} \simeq k$ induced by the p^nth-power map, we get an isomorphism of schemes $k^{1/p^n} \otimes_k H \simeq H^{(p^n)}$ (even an isomorphism of group schemes over the Frobenius isomorphism $k^{1/p^n} \simeq k$). Over the perfect closure k_{p} of k, the underlying reduced scheme of any finite type group scheme is smooth and hence is a subgroup scheme. Likewise, a smooth connected affine k_{p}-group has its radical (over $\overline{k}$) defined over k_{p}, by Galois descent. By expressing k_{p} as the direct limit of the extensions k^{1/p^n} of k, we thereby get some $n \geqslant 0$ such that the geometric radical of $k^{1/p^n} \otimes_k \overline{G}$ is defined over k^{1/p^n}, so by passing to $G^{(p^n)}$ and $H^{(p^n)}$ (and $\overline{G}^{(p^n)}$) for such n we can assume that $\overline{G}$ has its radical (over $\overline{k}$) defined over k. We can replace H with the pullback in G of the radical of $\overline{G}$ without affecting the solvability of $(H_{\overline{k}})^0_{\mathrm{red}}$, so in this way we can arrange that $\overline{G}$ is semisimple. There exists $n \geqslant 0$ such that $k^{1/p^n} \otimes_k H$ has underlying reduced scheme that is a smooth k^{1/p^n}-subgroup scheme, so the underlying reduced scheme of $H^{(p^n)}$ is a k-smooth subgroup scheme. Hence, by passing to $G^{(p^n)}$ and $H^{(p^n)}$ for such n we may assume that H_{red} is a smooth k-subgroup of H.

Since $G/H_{\mathrm{red}} \to G/H$ is a radiciel surjective homomorphism, by Lemma A.2.3 we can replace H by H_{red} to reduce to the case when H is smooth (so H^0 is solvable) but now $\overline{G} = G/H$ may not be smooth; however, $(\overline{G}_{\overline{k}})_{\mathrm{red}}$ is smooth and semisimple. A solvable smooth connected affine group over the perfect closure k_{p} has a (necessarily k_{p}-split) smooth connected unipotent normal k_{p}-subgroup modulo which it is a torus. Thus, by repeating the same direct limit and base change argument as was used above, we may use further Frobenius base change and descent from the perfect closure to get to the case when H has a *k-split* smooth connected unipotent normal k-subgroup U such that H^0/U a torus. This is the same property of H that we noted (at the end of §A.1) is automatically satisfied in the number field case.

To reduce to the case when G is smooth, argue as follows. Choose $n \geqslant 0$ such that $G_n := G/\ker F_{G/k,n}$ is k-smooth. (Of course, $\overline{G}_n := \overline{G}/\ker F_{\overline{G}/k,n}$ is then k-smooth, and even connected semisimple in view of our preceding reduction steps.) The surjective homomorphism $G_n \to \overline{G}_n$ is faithfully flat since the source and target are smooth, so it expresses $\overline{G}_n$ as the quotient of G_n modulo a closed normal subgroup scheme whose $\overline{k}$-fiber has underlying reduced subgroup that is a quotient of $(H_{\overline{k}})_{\rm red}$ by an infinitesimal normal subgroup and hence has solvable identity component. By Lemma A.2.3 applied to $G \to G_n$ and $\overline{G} \to \overline{G}_n$ it therefore suffices to prove Theorem A.1.1 when G is smooth and $\overline{G}$ is smooth and semisimple.

Going back through the preceding reduction steps concerning H (avoiding any changes to G beyond extension of the base field, and preserving smoothness and semisimplicity of $\overline{G}$) now brings us to the case when G and H are both smooth, $\overline{G}$ is semisimple, and H contains a k-split smooth connected unipotent normal k-subgroup U such that H^0/U is a torus.

A.3 Arguments with reductive groups

We have now reduced to considering the common setup in all characteristics as at the end of §A.1 in characteristic 0 and §A.2 in nonzero characteristic. Since H is normal in G, and $U_{\overline{k}}$ must be the unipotent radical of $H^0_{\overline{k}}$, the group U is normal in G because G is smooth. But U is a k-split smooth connected unipotent k-group, so it admits a composition series over k with successive quotients equal to $\mathbf{G}_{\rm a}$. Thus, Lemma A.2.1 gives that $G(k)\backslash G(\mathbf{A}_k)$ is topologically fibered over $(G/U)(k)\backslash(G/U)(\mathbf{A}_k)$ with fibers homeomorphic to $U(k)\backslash U(\mathbf{A}_k)$, and $U(k)\backslash U(\mathbf{A}_k)$ is compact by [Oes, IV, 1.3]. Hence,

$$G(k)\backslash G(\mathbf{A}_k) \to (G/U)(k)\backslash(G/U)(\mathbf{A}_k)$$

is proper. We may therefore replace G with G/U to reduce to the case when H^0 is a torus. The normality of H in G implies that of H^0 in G, and G/H^0 is semisimple since $\overline{G} = G/H$ is semisimple. Since G is connected and the automorphism functor of a torus is represented by an étale group, H^0 is in the center of G.

The factorization

$$G \to G/H^0 \to G/H$$

reduces us to separately treating the cases when H is a torus and when H is finite étale, with $\overline{G}$ connected and semisimple. Thus, we now assume that H is a torus (hence maximal central in G, as $\overline{G}$ is semisimple), and in §A.4 we treat the case when H is étale.

Let k'/k be a finite separable extension that splits the central torus H. Consider the central pushout $G \to \widetilde{G}$ of G by the canonical closed immersion $H \to {\rm R}_{k'/k}(H_{k'}) =: Z$. We have $G/H = \widetilde{G}/Z$, and the map $G \to \widetilde{G}$ is a closed immersion. Lemma 4.2.5 ensures that $G(k)\backslash G(\mathbf{A}_k)^1 \to \widetilde{G}(k)\backslash\widetilde{G}(\mathbf{A}_k)^1$ is a closed embedding, so we can replace $H \to G$ with $Z \to \widetilde{G}$. That is, we are reduced to the case when H is a power of ${\rm R}_{k'/k}({\rm GL}_1)$ for some finite separable extension k'/k. Hence, by (the proof of) Lemma A.2.1, the map

$$G(k)\backslash G(\mathbf{A}_k) \to \overline{G}(k)\backslash\overline{G}(\mathbf{A}_k) \tag{A.3.1}$$

is a fibration whose fibers are orbits for the continuous free right action of $H(k)\backslash H(\mathbf{A}_k)$ on $G(k)\backslash G(\mathbf{A}_k)$. In particular, the map (A.3.1) has continuous local cross-sections.

We now separately treat the cases of number fields and function fields, due to the different structure of the idelic norm and idelic topology in the two cases. The case $H = 1$ is trivial, so we can assume $H \neq 1$.

The following argument was explained to me by G. Prasad in the number field case, and it will be easily adapted to the function field case. Suppose that k is a number field. We will show that the local cross-sections to the central fibration $G(\mathbf{A}_k) \to \overline{G}(\mathbf{A}_k)$ can be chosen to land inside of $G(\mathbf{A}_k)^1$. This will provide local cross-sections to the natural map $\pi : G(k)\backslash G(\mathbf{A}_k)^1 \to \overline{G}(k)\backslash\overline{G}(\mathbf{A}_k)$, showing that π is a fibration whose fibers are orbits for the continuous free action of $H(k)\backslash H(\mathbf{A}_k)^1$. (We have $H(\mathbf{A}_k)^1 = G(\mathbf{A}_k)^1 \cap H(\mathbf{A}_k)$ since H is the identity component of the "reduced center" of the connected reductive k-group G.) This latter quotient is compact since it is a power of the norm-1 subgroup of $k'^{\times}\backslash\mathbf{A}_{k'}^{\times}$, so by Theorem 4.2.1 the properness of π will then follow. To build local cross-sections landing in $G(\mathbf{A}_k)^1$, it suffices to construct a continuous map of topological spaces $c : G(\mathbf{A}_k) \to H(\mathbf{A}_k)$ such that $c(g)^{-1}g \in G(\mathbf{A}_k)^1$ for all g.

Let $Z \subseteq H$ be the maximal k-split subtorus of the central k-torus H. Clearly $Z \neq 1$, due to the description of H in terms of Weil restrictions and the hypothesis $H \neq 1$. Since H is the identity component of the reduced center of the connected reductive G, Z is the maximal k-split central k-torus in G. By the structure of reductive groups, $\mathrm{X}_k(G)$ maps isomorphically onto a finite-index subgroup of $\mathrm{X}_k(Z)$. Thus, we can choose a basis $\chi_1, \dots, \chi_r$ of $\mathrm{X}_k(Z)$ such that $\chi_1^{e_1}, \dots, \chi_r^{e_r}$ is a basis of $\mathrm{X}_k(G)$. We use the χ_j's to define a k-isomorphism $Z \simeq \mathrm{GL}_1^r$, so upon choosing an archimedean place $v \in S$ we get a closed embedding

$$t : (\mathbf{R}_{>0}^{\times})^r \hookrightarrow (k_v^{\times})^r = Z(k_v) \hookrightarrow Z(\mathbf{A}_k) \subseteq H(\mathbf{A}_k)$$

via the canonical inclusion $\mathbf{R}_{>0}^{\times} \hookrightarrow k_v^{\times}$ for the archimedean place v. For any $g \in G(\mathbf{A}_k)$, define

$$c(g) = t(\|\chi_1^{e_1}(g)\|_k^{1/e_1}, \cdots, \|\chi_r^{e_r}(g)\|_k^{1/e_r}).$$

It is clear that c has the desired property, due to unique divisibility of $\mathbf{R}_{>0}^{\times}$ and how the χ_j's were chosen. This settles the case of number fields.

Next, suppose k is a global function field with constant field of size q, so the idelic norm on $\mathbf{A}_k^{\times}$ has image $Q = q^{\mathbf{Z}}$ in $\mathbf{R}_{>0}^{\times}$. Once again, let $Z \subseteq H$ be the maximal k-split subtorus and let $\{\chi_1, \dots, \chi_r\}$ be a basis of $\mathrm{X}_k(Z)$ such that $\{\chi_1^{e_1}, \dots, \chi_r^{e_r}\}$ is a basis of the finite-index image of $\mathrm{X}_k(G)$. The continuous homomorphism

$$\Phi = (\|\chi_1^{e_1}\|_k, \dots, \|\chi_r^{e_r}\|_k) : G(\mathbf{A}_k) \to Q^{\oplus r} \subseteq (\mathbf{R}_{>0}^{\times})^{\oplus r}$$

has image equal to a subgroup $\Gamma \subseteq Q^{\oplus r}$ (even of finite index, though we do not use this fact), and the restriction of this map to $H(\mathbf{A}_k)$ has image that we denote as $\Lambda \subseteq \Gamma$. For the maximal quotient map $\pi : G \twoheadrightarrow T$ onto a k-split torus, the restriction of π to the maximal k-split central torus Z in G is an isogeny since G is connected reductive. Thus, there is a map between these tori in the other direction such that their composite is multiplication on T by some nonzero integer. Hence, Γ/Λ is killed by this nonzero integer and so Λ has finite index in Γ.

Since $G(k) \to \overline{G}(k)$ is surjective and $G(k) \subseteq G(\mathbf{A}_k)^1$, for each $\overline{g} \in \overline{G}(\mathbf{A}_k)$ all elements $g \in G(\mathbf{A}_k)$ mapping to $\overline{G}(k) \cdot \overline{g}$ in $\overline{G}(k)\backslash\overline{G}(\mathbf{A}_k)$ give rise to the same left coset $H(\mathbf{A}_k)G(\mathbf{A}_k)^1 g$. Thus, by local constancy of the idelic norm we arrive at a natural decomposition of $\overline{G}(k)\backslash\overline{G}(\mathbf{A}_k)$ into open and closed subsets Y_j labelled by the finitely many elements of Γ/Λ; we enumerate the finite set Γ/Λ as $\{\overline{\gamma}_j\}$. For each j, let $g_j \in G(\mathbf{A}_k)$ be an element whose image in Γ/Λ is $\overline{\gamma}_j$, and take $g_{j_0} = 1$ for the unique j_0 such that $\overline{\gamma}_{j_0} = 1$. Thus, we get a finite disjoint union decomposition

$$G(k)\backslash G(\mathbf{A}_k) = \coprod G(k)\backslash H(\mathbf{A}_k)G(\mathbf{A}_k)^1 g_j$$

into the open and closed preimages of the Y_j's. Define

$$E = \coprod G(\mathbf{A}_k)^1 g_j = \coprod g_j G(\mathbf{A}_k)^1;$$

this is an open and closed set in $G(\mathbf{A}_k)$ that contains $G(\mathbf{A}_k)^1$ (since $g_{j_0} = 1$) and is stable under left and right translations by the normal subgroup $G(\mathbf{A}_k)^1$.

The topological group $G(\mathbf{A}_k)$ has a base of compact open sets, so on the open and closed subgroup

$$\Phi^{-1}(\Lambda) = H(\mathbf{A}_k)G(\mathbf{A}_k)^1 \subseteq G(\mathbf{A}_k)$$

it is trivial to construct a (locally constant) continuous map $c : \Phi^{-1}(\Lambda) \to H(\mathbf{A}_k)$ such that $c(g)^{-1}g \in G(\mathbf{A}_k)^1$ for all $g \in \Phi^{-1}(\Lambda)$. By using both the continuous map

$$g \mapsto c(gg_j^{-1})^{-1}g \in G(\mathbf{A}_k)^1 g_j$$

on each $H(\mathbf{A}_k)G(\mathbf{A}_k)^1 g_j$ and the local cross-sections to $G(\mathbf{A}_k) \to \overline{G}(\mathbf{A}_k)$, we see that the restriction of the $H(k)\backslash H(\mathbf{A}_k)$-equivariant fibration

$$G(k)\backslash G(\mathbf{A}_k) \to \overline{G}(k)\backslash \overline{G}(\mathbf{A}_k)$$

to the $H(k)\backslash H(\mathbf{A}_k)^1$-stable open and closed set $G(k)\backslash E$ admits local cross-sections. The resulting map

$$\pi_E : G(k)\backslash E \to \overline{G}(k)\backslash \overline{G}(\mathbf{A}_k)$$

must therefore be an $H(k)\backslash H(\mathbf{A}_k)^1$-equivariant fibration. (The topology is easy to analyze because E is open and closed in $G(\mathbf{A}_k)$ and we are using quotients by discrete subgroups.)

The continuous free action of $H(k)\backslash H(\mathbf{A}_k)^1$ on fibers of π_E is transitive. Thus, the fibration π_E has all fibers homeomorphic to $H(k)\backslash H(\mathbf{A}_k)^1$. Since $H(k)\backslash H(\mathbf{A}_k)^1$ is compact (argue as in the number field case), the map π_E is therefore a fibration with compact fibers and thus is proper. The restriction π of π_E to the closed subset $G(k)\backslash G(\mathbf{A}_k)^1$ is therefore also proper, as desired.

A.4 Cohomological arguments with étale H

In this section we finish the proof of Theorem A.1.1 by treating the case when H is étale. The intervention of $G(\mathbf{A}_k)^1$ can be removed, as we prove the following stronger result:

Proposition A.4.1. *Let G be an affine group scheme of finite type over a global field k, and H a normal étale closed k-subgroup. For $\overline{G} = G/H$, the natural map of topological spaces*

$$G(k)\backslash G(\mathbf{A}_k) \to \overline{G}(k)\backslash \overline{G}(\mathbf{A}_k)$$

is proper.

The applications in this paper only need the case of étale multiplicative H (for which there are simpler arguments). I am grateful to O. Gabber for showing me how to go beyond the étale multiplicative case.

Proof. Consider the maximal smooth k-subgroup $G' \subseteq G$ as in Lemma 3.1.1 (so $H \subseteq G'$). By the construction of G' and the smoothness of H, the image of G' in $\overline{G}$ is the maximal smooth k-subgroup of $\overline{G}$. This maximal smooth k-subgroup also contains all adelic points, so we may replace G with G' to arrange that G is smooth. Define $C = H \cap G^0$, so C is a finite étale normal k-subgroup of the smooth connected G^0. Thus, C is *central* in G^0. Clearly $\overline{G}^0 = G^0/C$. In the commutative square

$$\begin{array}{ccc} G^0(k)\backslash G(\mathbf{A}_k) & \longrightarrow & \overline{G}^0(k)\backslash \overline{G}(\mathbf{A}_k) \\ \downarrow & & \downarrow \\ G(k)\backslash G(\mathbf{A}_k) & \longrightarrow & \overline{G}(k)\backslash \overline{G}(\mathbf{A}_k) \end{array}$$

the vertical maps are finite covering spaces, so properness along the bottom is reduced to properness along the top. But $G/C \to \overline{G}$ is a proper map between separated k-schemes of finite type, so $(G/C)(\mathbf{A}_k) \to \overline{G}(\mathbf{A}_k)$ is topologically proper and hence

$$\overline{G}^0(k)\backslash(G/C)(\mathbf{A}_k) \to \overline{G}^0(k)\backslash\overline{G}(\mathbf{A}_k)$$

is proper. Thus, it suffices to prove that

$$G^0(k)\backslash G(\mathbf{A}_k) \to (G/C)^0(k)\backslash(G/C)(\mathbf{A}_k)$$

is proper. We can therefore replace H with C to reduce to the case that $H \subset G^0$ (so H is central in G^0), and in this case it suffices to prove the properness of

$$G^0(k)\backslash G(\mathbf{A}_k) \to \overline{G}^0(k)\backslash\overline{G}(\mathbf{A}_k).$$

Since $\pi : G \to \overline{G}$ is a proper map between separated k-schemes of finite type, the map $G(\mathbf{A}_k) \to \overline{G}(\mathbf{A}_k)$ is topologically proper. Defining

$$N := \mathrm{im}(G^0(k) \to \overline{G}^0(k)) \simeq \ker(\mathrm{H}^1(k, H) \to \mathrm{H}^1(k, G^0)),$$

the induced map

$$G^0(k)\backslash G(\mathbf{A}_k) \to N\backslash\overline{G}(\mathbf{A}_k) \tag{A.4.1}$$

is proper since its topological pullback by the quotient mapping $\overline{G}(\mathbf{A}_k) \to N\backslash\overline{G}(\mathbf{A}_k)$ is the natural map $(G^0(k)\cap H(k))\backslash G(\mathbf{A}_k) \to \overline{G}(\mathbf{A}_k)$ that is proper due to the finiteness of $G^0(k)\cap H(k)$. Since G^0 is a central extension of $\overline{G}^0$ by H, N is a normal subgroup of $\overline{G}^0(k)$ such that $\overline{G}^0(k)/N$ commutative (and even a subgroup of $\mathrm{H}^1(k, H)$). Any compact subset of $\overline{G}^0(k)\backslash\overline{G}(\mathbf{A}_k)$ has the form $\overline{K} := \overline{G}^0(k)\backslash\overline{G}^0(k)K$ for a compact subset K in $\overline{G}(\mathbf{A}_k)$. Its closed preimage in $G^0(k)\backslash G(\mathbf{A}_k)$ consists of the $G^0(k)$-cosets of points $g \in G(\mathbf{A}_k)$ such that $\pi(g) \in \overline{G}^0(k)K$.

Consider the subset $Y \subseteq \overline{G}^0(k)$ of points $y \in \overline{G}^0(k)$ such that yK meets $\pi(G(\mathbf{A}_k))$. Note that Y is stable under left multiplication by N. To prove that the closed preimage $\mathscr{X}$ of $\overline{K}$ in $G^0(k)\backslash G(\mathbf{A}_k)$ is compact, by the properness of (A.4.1) it is equivalent to prove that $\mathscr{X}$ maps onto a compact set of $N\backslash\overline{G}(\mathbf{A}_k)$. The image of $\mathscr{X}$ in $N\backslash\overline{G}(\mathbf{A}_k)$ consists of the N-cosets of points in the non-empty compact overlaps $yK \cap \pi(G(\mathbf{A}_k))$ for $y \in Y$. (Note that these compact overlaps are stable under left multiplication by N.) Provided that there are only finitely many such N-cosets of points $y \in Y$, we will be done.

A collection of N-cosets in $\overline{G}^0(k)$ is finite precisely when its image in $\mathrm{H}^1(k, H)$ is finite, so the problem is reduced to proving that the connecting map

$$\delta : \overline{G}(k) \to \mathrm{H}^1(k, H)$$

carries Y onto a finite set. We claim that finiteness of a subset of $\mathrm{H}^1(k, H)$ can be detected using "strictly local" methods:

Lemma A.4.2. *Let M be a finite discrete $\mathrm{Gal}(k_s/k)$-module. For a non-archimedean place v of k, let k_v^{sh} denote the fraction field of a strict henselization of $\mathscr{O}_v$. The fibers of the localization homomorphism*

$$\mathrm{H}^1(k, M) \to \prod_{v\nmid\infty} \mathrm{H}^1(k_v^{\mathrm{sh}}, M)$$

are finite, as are the fibers of $\mathrm{H}^1(k_v, M) \to \mathrm{H}^1(k_v^{\mathrm{sh}}, M)$ for each $v \nmid \infty$.

Proof. Let Σ be a finite set of places of k, containing the archimedean places in the number field case and non-empty in the function field case, such that M extends to a finite étale commutative

$\mathscr{O}_{k,\Sigma}$-group $\mathscr{M}$. By arguing in terms of unramified descent and finite étale torsors, we see that the kernel of the localization map is contained in $\mathrm{H}^1(\mathscr{O}_{k,\Sigma}, \mathscr{M})$, and this is finite in the number field case. The local assertion at the end of the lemma is obvious in characteristic 0, as even $\mathrm{H}^1(k_v, M)$ is finite in such cases.

Consider the global function field case. Let X be the smooth proper and geometrically connected curve over a finite field κ such that the function field of X is k, and let $j : U \hookrightarrow X$ be the dense open subscheme corresponding to $\operatorname{Spec} \mathscr{O}_{k,\Sigma}$. Then $\mathscr{M}$ corresponds to a locally constant constructible abelian sheaf $\mathscr{F}$ on $U_{\text{ét}}$, and arguing in terms of finite étale torsors identifies the kernel of the localization map with $\mathrm{H}^1_{\text{ét}}(X, j_*\mathscr{F})$. The pushforward $\mathscr{G} := j_*\mathscr{F}$ is a constructible abelian sheaf on $X_{\text{ét}}$ that is represented by the maximal X-étale open subscheme of the normalization of X in $\mathscr{M}$. The constructibility of $\mathscr{G}$ and the properness of the curve X ensure that $\mathrm{H}^m_{\text{ét}}(X_{\kappa_s}, \mathscr{G}_{\kappa_s})$ is finite for all $m \geqslant 0$. In view of the finiteness properties of Galois cohomology of a finite field, the Leray spectral sequence

$$\mathrm{H}^n(\kappa, \mathrm{H}^m_{\text{ét}}(X_{\kappa_s}, \mathscr{G}_{\kappa_s})) \Rightarrow \mathrm{H}^{n+m}_{\text{ét}}(X, \mathscr{G})$$

implies that $\mathrm{H}^1_{\text{ét}}(X, j_*\mathscr{F})$ is finite.

In the local function field case, the finiteness assertion at the end of the lemma amounts to the finiteness of $\mathrm{H}^1(\mathscr{O}_v, \mathscr{G})$ for any (necessarily constructible) étale abelian sheaf $\mathscr{G}$ on $\operatorname{Spec} \mathscr{O}_v$ that is the pushforward of the étale sheaf on $\operatorname{Spec} k_v$ corresponding to a finite Galois module. Such a pushforward is represented by a quasi-finite separated étale commutative group scheme whose "finite part" (in the sense of [EGA, IV_4, 18.5.11(c)] with $S = \operatorname{Spec}(\mathscr{O}_v)$) is an open and closed $\mathscr{O}_v$-subgroup filling up the special fiber, so it suffices to separately prove the desired finiteness in the cases when $\mathscr{G}$ is either (i) finite étale or (ii) has vanishing special fiber. In case (ii) the degree-1 cohomology vanishes, as the corresponding torsors admit *unique* $\mathscr{O}_v$-points (due to unramified descent of the obvious analogous property over $\mathscr{O}_v^{\mathrm{sh}}$). In case (i), the finiteness of $\mathrm{H}^1(\mathscr{O}_v, \mathscr{G})$ is due to the fact that the corresponding torsors split over a finite unramified extension of degree bounded by $r_{\mathscr{G}}!$, where $r_{\mathscr{G}}$ is the fiber-rank of $\mathscr{G}$. ∎

By Lemma A.4.2, to prove that Y has finite image in $\mathrm{H}^1(k, H)$ it suffices to prove its image in $\mathrm{H}^1(k_v^{\mathrm{sh}}, H)$ is finite for all non-archimedean v and vanishes for all but finitely many such v. The condition $yq = \pi(g)$ for some $g \in G(\mathbf{A}_k)$ and $q \in K \subseteq \overline{G}(\mathbf{A}_k)$ implies (by [Se2, I, §5.6, Cor. 1]) that

$$0 = \delta_v(y_v q_v) = \delta_v(q_v) + q_v^{-1}.\delta_v(y_v)$$

for all non-archimedean places v of k, where we use the natural action of $\overline{G}(k_v)/\pi(G^0(k_v))$ on $\mathrm{H}^1(k_v, H)$. (Note that H may not be central in $G(k_v)$, but it is central in $G^0(k_v)$.) Thus, if K_v denotes the compact image of K under the projection $\overline{G}(\mathbf{A}_k) \to \overline{G}(k_v)$ then $\delta_v(Y)$ has vanishing image in $\mathrm{H}^1(k_v^{\mathrm{sh}}, H)$ if the same holds for $\delta_v(K_v)$, and $\delta_v(Y)$ has finite image in $\mathrm{H}^1(k_v^{\mathrm{sh}}, H)$ if the same holds for $\delta_v(K_v)$ (since the natural $\overline{G}(k_v)$-action on $\mathrm{H}^1(k_v, H)$ is through the discrete quotient $\overline{G}(k_v)/\pi(G^0(k_v))$ in which the compact K_v^{-1} must have finite image).

It remains to prove that the image of $\delta_v(K_v)$ in $\mathrm{H}^1(k_v^{\mathrm{sh}}, H)$ vanishes for all but finitely many v and is finite for all v. The image $\pi(G^0(k_v)) \subseteq \overline{G}(k_v)$ is an open subgroup, so the compact K_v is covered by finitely many $\pi(G^0(k_v))$-cosets inside of $\overline{G}(k_v)$. Hence, the finiteness of $\delta_v(K_v)$ for all v is clear. To get the vanishing for all but finitely many such v, we pick a finite non-empty set S of places of k such that G extends to a smooth affine $\mathscr{O}_{k,S}$-group G_S in which H extends to a finite étale closed normal subgroup H_S. Then $\overline{G}_S := G_S/H_S$ is smooth and affine with generic fiber $\overline{G}$, so $G_S(\mathscr{O}_v^{\mathrm{sh}}) \to \overline{G}_S(\mathscr{O}_v^{\mathrm{sh}})$ is surjective for all $v \notin S$. By Weil's description of the topology on $\overline{G}(\mathbf{A}_k)$, there exists a finite set S' containing S such that

$$K \subseteq \prod_{v \in S'} \overline{G}(k_v) \times \prod_{v \notin S'} \overline{G}_S(\mathscr{O}_v).$$

Hence, if $v \notin S'$ then K_v is in the image of $G(k_v^{\mathrm{sh}}) \to \overline{G}(k_v^{\mathrm{sh}})$, so $\delta_v(K_v)$ has vanishing image in $\mathrm{H}^1(k_v^{\mathrm{sh}}, H)$. ■

A.5 An application to compactness

Let G be a smooth connected affine group over a global field k. It was conjectured by Godement (at least if $\mathrm{char}(k) = 0$) that $G(k)\backslash G(\mathbf{A}_k)^1$ is compact if and only if the following two conditions both hold:

(a) every k-split torus $T \subseteq G$ satisfies $T_{\overline{k}} \subseteq \mathscr{R}(G_{\overline{k}})$,

(b) every k-split smooth connected unipotent k-subgroup $U \subseteq G$ satisfies $U_{\overline{k}} \subseteq \mathscr{R}(G_{\overline{k}})$.

It was announced without proof by Borel and Tits that (a) and (b) are equivalent over any field; a proof is given in Proposition A.5.1 below. Obviously (a) and (b) always hold when G is solvable, and the compactness of $G(k)\backslash G(\mathbf{A}_k)^1$ was proved in general for solvable G by Godement–Oesterlé [Oes, IV, 1.3]. In the number field case, Godement's conjecture was proved independently by Borel and Harish-Chandra [BHC, Thm. 11.8] (see [Bo1, Thm. 5.8] for a treatment in adelic terms) and by Mostow and Tamagawa [MT, II, §3, Thm.]. Over global function fields the same assertion for reductive G is due to Harder (Theorem 5.1.1(ii)).

In [Oes, IV, 1.4] it is shown that in the function field case the compactness criterion (a) is necessary in general, but the sufficiency is proved there under restrictive hypotheses on the field of definition of the geometric radical. The key missing ingredient for avoiding such hypotheses is the structure theory for pseudo-reductive groups as in §2.3. Using that structure theory and Theorem A.1.1, below we prove the sufficiency of (a) in general (i.e., with no hypotheses on the geometric radical). Our proof uses the global result in Theorem 5.1.1(i) that is not applicable over number fields, so it does not give a new proof of the sufficiency of Godement's criterion (a) in the number field case.

Proposition A.5.1. *The conditions* (a) *and* (b) *above are equivalent over an arbitrary field k.*

Proof. The case of perfect k is easier, but we give a uniform argument over all fields. The first step is to give a formulation of conditions (a) and (b) directly over k (i.e., without the intervention of an algebraic closure) by working with certain quotients of G over k. Let $\overline{G}$ denote the maximal pseudo-reductive quotient $G/\mathscr{R}_{u,k}(G)$ over k.

Lemma A.5.2. *Property* (a) *for G is equivalent to the condition that every k-split torus in $\overline{G}$ is central.*

Proof. The sufficiency is clear, and for necessity suppose that $T_{\overline{k}} \subseteq \mathscr{R}(G_{\overline{k}})$ for every k-split torus T in G. Let $\overline{T}$ be a k-split torus in $\overline{G}$, so by the smoothness of $G \twoheadrightarrow \overline{G}$ we can lift $\overline{T}$ to a k-split torus T in G. Property (a) for G implies that T is contained in the k-radical $\mathscr{R}_k(G)$ (i.e., the maximal solvable smooth connected normal k-subgroup of G), so likewise $\overline{T} \subseteq \mathscr{R}_k(\overline{G})$. But $\mathscr{R}_k(\overline{G})$ is pseudo-reductive since $\overline{G}$ is pseudo-reductive, so by solvability it is commutative [CGP, Prop. 1.2.3]. Thus, $\mathscr{R}_k(\overline{G})$ contains a unique maximal k-torus and this torus is central in $\overline{G}$ since $\overline{G}$ is smooth and connected. In particular, $\overline{T}$ is central in $\overline{G}$, as desired. ■

Lemma A.5.3. *Property* (b) *for* G *is equivalent to the condition that* $\overline{G}$ *contains no nontrivial* k*-split smooth connected unipotent* k*-subgroup.*

Proof. Sufficiency is once again obvious. For necessity, assume that G satisfies (b). Let $\mathscr{R}_{us,k}(G)$ be the maximal k-split smooth connected unipotent normal k-subgroup of G. This is the maximal k-split smooth connected normal k-subgroup of $\mathscr{R}_{u,k}(G)$ [CGP, Cor. B.3.5], so $\mathscr{R}_{u,k}(G)/\mathscr{R}_{us,k}(G)$ is k-wound in the sense of Definition 7.1.1ff. Since every k-split smooth connected unipotent k-subgroup U in G lies in $\mathscr{R}_u(G_{\overline{k}})$ by the hypothesis (b), it follows from [CGP, Lemma 1.2.1] (applied to the pseudo-reductive $\overline{G}$) that $U \subseteq \mathscr{R}_{u,k}(G)$ and hence $U \subseteq \mathscr{R}_{us,k}(G)$.

We may now rename $G/\mathscr{R}_{us,k}(G)$ as G to reduce to showing that if G contains no nontrivial k-split smooth connected unipotent k-subgroup then the same holds for $\overline{G}$. This is obvious if k is perfect (as then $\mathscr{R}_{u,k}(G) = \mathscr{R}_{us,k}(G) = 1$, forcing $\overline{G} = G$), so the real content is the case of imperfect k. (The subtlety for imperfect k is due to the fact that there exists k-wound smooth connected unipotent k-groups U admitting a *smooth connected* normal k-subgroup U' such that $U/U' = \mathbf{G}_{\rm a}$ [Oes, V, 3.5].)

We claim that G cannot contain any non-central k-split torus, or equivalently that every k-homomorphism $\lambda : \mathrm{GL}_1 \to G$ is central. Our proof will rely on the associated closed k-subgroup schemes $Z_G(\lambda)$ and $U_G(\pm\lambda)$ from [CGP, Lemma 2.1.5] that are normalized by λ. The k-group $Z_G(\lambda)$ is the scheme-theoretic centralizer of λ, and the k-groups $U_G(\pm\lambda)$ are unipotent and normalized by $Z_G(\lambda)$. By [CGP, Prop. 2.1.8] these k-subgroups are smooth and connected, and the multiplication map

$$U_G(-\lambda) \times Z_G(\lambda) \times U_G(\lambda) \to G$$

is an open immersion. The k-groups $U_G(\pm\lambda)$ are k-split [CGP, Prop. 2.1.10], hence trivial, so $G = Z_G(\lambda)$. This proves the desired centrality of λ in G.

Next, we need to use some elementary facts from theory of pseudo-parabolic k-subgroups, developed in [CGP, §2.2]. The centrality of all k-homomorphisms $\lambda : \mathrm{GL}_1 \to G$ implies that the only pseudo-parabolic k-subgroup of G (in the sense of [CGP, Def. 2.2.1]) is the entire group G. Hence, by [CGP, Lemma 2.2.3], the pseudo-reductive $\overline{G}$ contains no non-central k-split torus and thus contains no proper pseudo-parabolic k-subgroup. But [CGP, Thm. C.3.8] implies that any k-split smooth connected unipotent k-subgroup $\overline{U}$ in $\overline{G}$ is contained in $\mathscr{R}_{us,k}(\overline{P})$ for some pseudo-parabolic k-subgroup $\overline{P}$ in $\overline{G}$. Since necessarily $\overline{P} = \overline{G}$, we deduce that $\overline{U} = 1$, as desired. ■

Returning to the proof of Proposition A.5.1, by Lemma A.5.2 it follows that (a) for G is equivalent to (a) for $\overline{G}$, and Lemma A.5.3 implies the same for (b). Thus, we may assume G is pseudo-reductive and have to prove that G contains GL_1 as a non-central k-subgroup if and only if G contains $\mathbf{G}_{\rm a}$ as a k-subgroup. In the proof of Lemma A.5.3 we showed that if there is no $\mathbf{G}_{\rm a}$ as a k-subgroup then there is no non-central GL_1 as a k-subgroup. Conversely, assuming there is a k-subgroup $U \simeq \mathbf{G}_{\rm a}$ in G, we seek to construct a non-central GL_1 as a k-subgroup of G. By [CGP, Thm. C.3.8], there exists a pseudo-parabolic k-subgroup P in G such that $U \subseteq \mathscr{R}_{us,k}(P)$. In particular, $P \neq G$ since G is pseudo-reductive and $U \neq 1$. By definition $P := P_G(\lambda) = U_G(\lambda)Z_G(\lambda)$ for some k-homomorphism $\lambda : \mathrm{GL}_1 \to G$, due to the pseudo-reductivity of G. Since $P \neq G$, so $Z_G(\lambda) \neq G$, it follows that λ is non-central in G. ■

Remark A.5.4. Since torus centralizers in smooth connected affine groups H are connected and have the expected Lie algebra inside of $\mathrm{Lie}(H)$, by Proposition A.5.1 and Lemma A.5.2 we may reformulate Godement's compactness criterion as the condition that the maximal pseudo-reductive quotient of G over k has an empty associated relative root system.

Here is the general sufficiency of Godement's compactness criterion over global function fields:

Theorem A.5.5. *Let k be a global function field, and let G be a smooth connected affine k-group.*

(i) *If every k-split torus T in G satisfies $T_{\overline{k}} \subseteq \mathscr{R}(G_{\overline{k}})$ then $G(k)\backslash G(\mathbf{A}_k)^1$ is compact.*

(ii) *If an affine k-group scheme H of finite type does not contain* GL_1 *over k then $H(k)\backslash H(\mathbf{A}_k)$ is compact.*

Proof. Part (ii) is deduced from part (i) as follows. First suppose H is smooth and connected. We simply have to check that H has no nontrivial k-rational characters (as then $H(\mathbf{A}_k)^1 = H(\mathbf{A}_k)$, so we can conclude by (i)). If $\chi : H \to \mathrm{GL}_1$ is a nontrivial k-rational character then any maximal k-torus S in H must be carried onto GL_1, forcing S to be k-isotropic and hence contradicting the assumption that H does not contain GL_1. Next, suppose H is smooth but possibly disconnected. The compactness of the topological quotient group $H(\mathbf{A}_k)/H^0(\mathbf{A}_k)$ (Corollary 3.2.1) implies that $H(\mathbf{A}_k) = H^0(\mathbf{A}_k)K$ for a compact subset $K \subseteq H(\mathbf{A}_k)$. Since H^0 does not contain GL_1 as a k-subgroup (due to the same for H), $H^0(k)\backslash H^0(\mathbf{A}_k)$ is compact by the smooth connected case. Thus, $H^0(\mathbf{A}_k) = H^0(k)K'$ for a compact subset $K' \subseteq H^0(\mathbf{A}_k)$, so $H(\mathbf{A}_k) = H^0(k)K'K$. This proves the compactness of $H(k)\backslash H(\mathbf{A}_k)$ when H is smooth. In general, by Lemma 3.1.1 we can replace H with a suitable *smooth* closed k-subgroup without changing the topological group $H(\mathbf{A}_k)$ or its subgroup $H(k)$. This settles (ii) in general.

We may and do now restrict our attention to (i). First we reduce to the pseudo-reductive case, so let $\overline{G} := G/\mathscr{R}_{u,k}(G)$ be the maximal pseudo-reductive quotient of G over k. Every maximal k-split torus of $\overline{G}$ is the image of one of G, so the hypothesis on maximal k-split tori in G is inherited by $\overline{G}$. Also, since every k-rational character of G kills $\mathscr{R}_{u,k}(G)$, we see that the map $G(\mathbf{A}_k) \to \overline{G}(\mathbf{A}_k)$ carries $G(\mathbf{A}_k)^1$ into $\overline{G}(\mathbf{A}_k)^1$. Thus, we get a natural map $G(k)\backslash G(\mathbf{A}_k)^1 \to \overline{G}(k)\backslash \overline{G}(\mathbf{A}_k)^1$, and by Theorem A.1.1 this latter map is proper. We may therefore replace G with $\overline{G}$ to reduce to the case when G is pseudo-reductive over k. In particular, the torus hypothesis on G now says that all maximal k-split tori in G are central (so there is only one such torus).

If G is commutative then the compactness of $G(k)\backslash G(\mathbf{A}_k)^1$ is [Oes, IV, 1.3], so now assume G is non-commutative. By Theorem 2.3.6(ii) and Theorem 2.3.8 (including the triviality of the k-rational character group of the perfect k-group G_1 in Theorem 2.3.6(ii)), G is a generalized standard pseudo-reductive group. Let $(G', k'/k, T', C)$ be a generalized standard presentation adapted to a choice of maximal k-torus $\mathscr{T}$ in G (Remark 2.3.4), so C is the Cartan k-subgroup $Z_G(\mathscr{T})$ of G and

$$G \simeq (\mathrm{R}_{k'/k}(G') \rtimes C)/\,\mathrm{R}_{k'/k}(C') \tag{A.5.1}$$

with $C' = Z_{G'}(T')$.

Lemma A.5.6. *Let G be a non-commutative generalized standard pseudo-reductive group over a field k, and let $(G', k'/k)$ be the canonically associated pair underlying all generalized standard presentations of G. Godement's condition* (a) *is equivalent to the k'-anisotropicity of G'* (*i.e., each fiber of $G' \to \operatorname{Spec} k'$ is anisotropic*).

Proof. If $(G', k'/k, T', C)$ is the generalized standard presentation of G adapted to a maximal k-torus $\mathscr{T}$ in G then by [CGP, Prop. 10.2.2(2)] the k-torus $\mathscr{T}$ is the unique maximal one in G that contains the maximal k-torus of the commutative image of $\mathrm{R}_{k'/k}(C') \to C \hookrightarrow G$, and moreover $\mathscr{T} \mapsto T'$ is a bijection between the sets of maximal k-tori in G and maximal k'-tori in G'.

Now assume that $T_{\overline{k}} \subseteq \mathscr{R}(G_{\overline{k}})$ for every k-split torus T in G. We wish to prove that G' is k'-anisotropic. The explicit description of $\mathscr{T}$ given in [CGP, Prop. 10.2.2(2)] in terms of both T' and the maximal central k-torus in G implies that if G' is not k'-anisotropic then $\mathscr{D}(G)$ is k-isotropic. Since $\mathscr{D}(G)$ is perfect [CGP, Prop. 1.2.6], the identity component of the underlying reduced scheme U of $\mathscr{D}(G)_{\overline{k}} \cap \mathscr{R}(G_{\overline{k}})$ is unipotent. If there exists a nontrivial k-split torus S in $\mathscr{D}(G)$ then $S_{\overline{k}} \subseteq U$ by our hypothesis on k-split tori in G, which is absurd since U is unipotent. This proves that G' is k'-anisotropic when Godement's condition (a) holds.

Conversely, assume G' is k'-anisotropic, so $G'/Z_{G'}$ is as well and hence $\mathrm{R}_{k'/k}(C'/Z_{G'})$ is k-anisotropic. Thus, all k-split tori S in the Cartan k-subgroup C are killed by the map $C \to \mathrm{R}_{k'/k}(C'/Z_{G'})$ underlying the semidirect product in (A.5.1) and so all such S are central in G (due to how the semidirect product $\mathrm{R}_{k'/k}(G') \rtimes C$ is defined). By [CGP, Prop. 10.2.2(3)], the generalized standard presentation of G may be chosen to rest on $(G', k'/k)$ and any choice of maximal k-torus of G (see Remark 2.3.4), or equivalently any choice of maximal k'-torus T' of G' or choice of Cartan k-subgroup of G. Thus, our conclusion about C given the k'-anisotropicity of G' applies to *every* Cartan k-subgroup of G. That is, the maximal k-split torus of each Cartan k-subgroup of G is central in G, so all k-split tori in G are central. ∎

By Lemma A.5.6, our problem is to prove that $G(k)\backslash G(\mathbf{A}_k)^1$ is compact when G' is k'-anisotropic. The pseudo-reductivity has served its purpose and will no longer be needed. What we will continue to use is the "generalized standard presentation" of G, so in other words the pseudo-reductivity of C will no longer be relevant.

We next reduce to the case when C is k-anisotropic (possibly losing the pseudo-reductive property in the process). The k'-anisotropicity hypothesis on G' implies that the unique maximal k-split torus T_0 in C has trivial image in $\mathrm{R}_{k'/k}(C'/Z_{G'})$ and hence is central in G, so it makes sense to consider the exact sequence

$$1 \to T_0 \to G \to G/T_0 \to 1.$$

The image $C_0 := C/T_0 \subseteq G/T_0$ of the *commutative* Cartan k-subgroup $C \subseteq G$ is the Cartan subgroup $Z_{G/T_0}(\mathscr{T}/T_0)$. In particular, C_0 is k-anisotropic; beware that C_0 may not be pseudo-reductive. It is obvious that G/T_0 has a "generalized standard presentation" essentially the same as that of G except that we replace C with C_0 (so G/T_0 is pseudo-reductive if and only if C_0 is pseudo-reductive), and since G/T_0 contains a k-anisotropic maximal k-torus it has no nontrivial k-rational characters. Thus, $(G/T_0)(\mathbf{A}_k)^1 = (G/T_0)(\mathbf{A}_k)$. By Theorem A.1.1, the natural map $G(k)\backslash G(\mathbf{A}_k)^1 \to (G/T_0)(k)\backslash (G/T_0)(\mathbf{A}_k)$ is proper. Hence, we may replace G with G/T_0 to reduce to the case when C is k-anisotropic at the expense of possibly losing pseudo-reductivity but retaining the "generalized standard" form. In this case we aim to prove that $G(k)\backslash G(\mathbf{A}_k)$ is compact.

Now we apply the technique from §5.2, namely the exact sequences (5.2.2) and (5.2.3) whose notation we freely use. As we have noted already, since G contains a k-anisotropic maximal k-torus, the k-rational character group $\mathrm{X}_k(G)$ is trivial. It is important to check that the restriction map $\mathrm{X}_k(\mathscr{E}) \to \mathrm{X}_k(\mathscr{Z})$ is an isomorphism. By definition $\mathscr{E} = (H \times \mathscr{Z})/Z$ with $H := \mathrm{R}_{k'/k}(G') \rtimes C$, so $\mathrm{X}_k(H) = 1$ since C is k-anisotropic and $\mathrm{R}_{k'/k}(G')$ is perfect (as each fiber of $G' \to \operatorname{Spec} k'$ is absolutely pseudo-simple and either simply connected semisimple or basic exotic, so its Weil restriction to k is perfect by [CGP, Prop. 8.1.2, Cor. A.7.11]). Thus, any k-rational character of $\mathscr{E}$ must arise from one of $\mathscr{Z}$ that is trivial on Z. But $Z := \mathrm{R}_{k'/k}(C')$ is k-anisotropic since C' is k'-anisotropic, so all k-rational characters of $\mathscr{Z}$ are trivial on Z. Hence, indeed $\mathrm{X}_k(\mathscr{E}) = \mathrm{X}_k(\mathscr{Z})$. It follows that $\mathscr{E}(\mathbf{A}_k)^1/\mathscr{Z}(\mathbf{A}_k)^1 = \mathscr{E}(\mathbf{A}_k)/\mathscr{Z}(\mathbf{A}_k)$, so the exactness of the sequence on $\mathbf{A}_k$-points induced by (5.2.3) (due to the cohomological triviality of $\mathscr{Z}$) implies that the sequence of abstract groups

$$1 \to \mathscr{Z}(\mathbf{A}_k)^1 \to \mathscr{E}(\mathbf{A}_k)^1 \to G(\mathbf{A}_k) \to 1$$

is exact. Since the induced sequence on k-points is also exact, the natural map $\mathscr{E}(k)\backslash\mathscr{E}(\mathbf{A}_k)^1 \to G(k)\backslash G(\mathbf{A}_k)$ is surjective. It now suffices to prove the compactness of $\mathscr{E}(k)\backslash\mathscr{E}(\mathbf{A}_k)^1$, and for this we shall use the exact sequence (5.2.2) whose leftmost term has trivial local and global degree-1 Galois cohomology (Theorem 5.1.1).

Let $\mathscr{C}$ denote the rightmost term in (5.2.2), so it is a smooth connected commutative affine k-group. The pullback mapping $\mathrm{X}_k(\mathscr{C}) \to \mathrm{X}_k(\mathscr{E})$ is an isomorphism since $\mathrm{R}_{k'/k}(G')$ has no nontrivial k-rational characters. Thus, the topological exactness of the sequence induced by (5.2.2) on $\mathbf{A}_k$-points and the discreteness of the idelic norm on $\mathbf{A}_k^\times$ imply the topological exactness of the sequence

$$1 \to \mathrm{R}_{k'/k}(G')(\mathbf{A}_k) \to \mathscr{E}(\mathbf{A}_k)^1 \to \mathscr{C}(\mathbf{A}_k)^1 \to 1.$$

We also have an induced exact sequence on k-points. By Lemma A.2.1 and the discreteness of the idelic norm in the function field case, the continuous map $\Pi : \mathscr{E}(k)\backslash\mathscr{E}(\mathbf{A}_k)^1 \to \mathscr{C}(k)\backslash\mathscr{C}(\mathbf{A}_k)^1$ is a fibration with all fibers topologically isomorphic to the space $\mathrm{R}_{k'/k}(G')(k)\backslash \mathrm{R}_{k'/k}(G')(\mathbf{A}_k) = \prod G'_i(k'_i)\backslash G'_i(\mathbf{A}_{k'_i})$ (where $\{k'_i\}$ is the set of factor fields of k' and G'_i is the k'_i-fiber of G'). This fiber space is compact by Theorem 5.1.1(ii) (since each G'_i is k'_i-anisotropic), so the fibration Π is proper. Thus, the compactness of $\mathscr{C}(k)\backslash\mathscr{C}(\mathbf{A}_k)^1$ (which follows from the settled commutative case) implies the desired compactness of $\mathscr{E}(k)\backslash\mathscr{E}(\mathbf{A}_k)^1$. ■

We end this section with a local analogue of Godement's global compactness criterion.

Proposition A.5.7. *Let G be a smooth connected affine group over a local field k. Then $G(k)$ is compact if and only if G contains neither* GL_1 *nor* $\mathbf{G}_{\mathrm{a}}$ *as k-subgroups.*

In the reductive case this is a well-known result (with an elementary proof in [Pr2]). Our proof in general will ultimately reduce to this case over finite extensions via the structure theory of pseudo-reductive groups.

Proof. The "only if" direction is obvious. For the converse, first note that $\mathscr{R}_{us,k}(G) = 1$, so $U := \mathscr{R}_{u,k}(G)$ is k-wound. (Obviously $U = 1$ if $\mathrm{char}(k) = 0$.) Hence, $U(k)$ is compact [Oes, VI, §1]. Since $G(k)/U(k)$ is naturally identified with an open subgroup of $(G/U)(k)$, it suffices to prove that $(G/U)(k)$ is compact. Since G trivially satisfies Godement's condition (b), by Lemma A.5.3 it follows that G/U does not contain $\mathbf{G}_{\mathrm{a}}$ as a k-subgroup. By Proposition 3.1.3 (or an elementary direct argument), the quotient G/U does not contain GL_1 as a k-subgroup since the same holds for G. Thus, G/U satisfies the initial hypotheses too, so we may and do now assume that G is pseudo-reductive over k.

First consider the commutative case, so there is a short exact sequence

$$1 \to T \to G \to U \to 1$$

with a k-anisotropic torus T and unipotent U. Since $T(k)$ is compact and $G(k)/T(k)$ is an open subgroup of $U(k)$, it suffices to prove that $U(k)$ is compact. But G does not contain $\mathbf{G}_{\mathrm{a}}$ as a k-subgroup, so the same holds for $U = G/T$ due to Lemma 4.1.4. Hence, U is k-wound ($U = 1$ if $\mathrm{char}(k) = 0$), so $U(k)$ is compact.

Now we may assume that G is non-commutative. The argument at the end of the proof of Proposition A.5.1 shows that in the pseudo-reductive case, the properties that GL_1 and $\mathbf{G}_{\mathrm{a}}$

do not arise as k-subgroups are equivalent. Thus, we may now focus just on the property that GL_1 is not a k-subgroup of G. By Theorem 2.3.6(ii) and Theorem 2.3.8, we may assume that G is a generalized standard pseudo-reductive group. (This reduction step uses Lemma 4.1.3.) Choose a maximal k-torus T in G, so for $C := Z_G(T)$ there is an associated isomorphism (A.5.1). The settled commutative pseudo-reductive case implies that $C(k)$ is compact. By [CGP, Prop. 10.2.2(2)], the k-anisotropicity of all maximal k-tori in G implies the same for all maximal k'-tori in G'. Hence, if $\{k'_i\}$ is the set of factor fields of k' and G'_i is the k'_i-fiber of G' then the group $\mathrm{R}_{k'/k}(G')(k) = G'(k') = \prod G'_i(k'_i)$ is compact by the known semisimple case and its analogue in the basic exotic case (which reduces to the semisimple case by Theorem 2.3.8(ii)).

The quotient on the right side of (A.5.1) is central, so the compact $(\mathrm{R}_{k'/k}(G') \rtimes C)(k)$ has image in $G(k)$ that is an open *normal* subgroup. It suffices to check that this subgroup has finite index. Such finiteness follows from that of $\mathrm{H}^1(k, \mathrm{R}_{k'/k}(C'))$ (see Lemma 4.1.6, as well as Proposition 4.1.7(i) and its trivial archimedean analogue; we implicitly use the identification of the cohomology of C' with that of a related k'-torus in the presence of basic exotic fibers for $G' \to \mathrm{Spec}(k')$ in characteristics 2 and 3, as explained in the proof of Proposition 4.1.9). ■

Appendix B. Twisting in flat cohomology via torsors

Consider a group scheme G of finite type over a field k. It is necessarily quasi-projective (see [CGP, Prop. A.3.5]). The pointed set $\mathrm{H}^1(k, G)$ of isomorphism classes of right G-torsors over k for the *fppf* topology is a functor via pushout $P \rightsquigarrow P \times^G G'$ along k-homomorphisms $G \to G'$ between k-group schemes of finite type. We now develop some theory for this functor, since most literature on it is written in tremendous generality (over ringed topoi, etc.) and omits a detailed discussion of the definitions and basic properties that we need.

The following discussion is a variant on [Se2, I, §5.3–§5.7], bypassing smoothness hypotheses on G. The case of smooth groups is sufficient for our needs except in two places: the reduction of Theorem 1.3.3 to the case of smooth G in §6.1 (see Remark 1.3.4) and the proof of Theorem 7.1.3. What we do below is consistent with the cocycle constructions in [Se2, I, §5.3–§5.7], but we do not need this consistency and so we will not address it here aside from some remarks. Two approaches can be used in the absence of smoothness: Isom-functors (cf. [GM, §2]) and concrete quotient constructions. We consider both points of view.

B.1 Inner forms of groups

Let Y be a right G-torsor over k. We shall use Y to define a k-form ${}_YG$ of G, called the *twist* of G by Y. Let ${}_YG$ denote the *fppf* sheaf quotient of $Y \times Y$ modulo the diagonal right action by G ($(y_1, y_2).g = (y_1.g, y_2.g)$). By descent from the case of split torsors over a finite extension of k, and the effectivity of descent through such finite extensions for quasi-projective schemes, this quotient is represented by a quasi-projective k-scheme.

Next, we endow ${}_YG$ with a k-group structure. For any points y_1 and y_2 of Y valued in a k-algebra R we write $[y_1, y_2]$ to denote the image of (y_1, y_2) in $({}_YG)(R)$. In many situations R will arise as a faithfully flat extension of a k-subalgebra R_0 and $[y_1, y_2]$ descends to R_0; we leave it to the interested reader to keep track of such descent issues when used implicitly below. As but one example, the diagonal points $(y, y) \in (Y \times Y)(R)$ for all R (when $Y(R)$ is non-empty) descend to a common point $e \in ({}_YG)(k)$ that we shall denote as $[y, y]$ even though $Y(k)$ is typically empty. Likewise, there is a unique well-defined associative composition law on the k-scheme ${}_YG$

determined by the requirement

$$[y, y.g_1] \cdot [y, y.g_2] = [y, y.(g_1 g_2)].$$

It is clear that the distinguished point e is a 2-sided identity for this composition law and that the flip involution on $Y \times Y$ induces an inverse on ${}_YG$ over k.

In this way ${}_YG$ is endowed with a structure of quasi-projective k-group scheme. Conceptually, ${}_YG$ represents the automorphism functor $\underline{\mathrm{Aut}}_G(Y)$ of the right G-torsor Y by assigning to any pair $(y_1, y_2) \in Y(R) \times Y(R)$ the unique G_R-automorphism of Y_R carrying y_1 to y_2. The formation of ${}_YG$ commutes with arbitrary scalar extension on k, and if Y is a trivial right G-torsor then any choice of $y_0 \in Y(k)$ defines an isomorphism of k-groups $G \simeq {}_YG$ via $g \mapsto [y_0, y_0.g]$. (Equivalently, this is the isomorphism $G \simeq \underline{\mathrm{Aut}}_G(Y)$ carrying g to the automorphism $y_0.g' \mapsto y_0.gg'$.) Thus, the k-groups ${}_YG$ and G become isomorphic over any extension K/k such that $Y(K)$ is non-empty, as is also clear via the isomorphism ${}_YG \simeq \underline{\mathrm{Aut}}_G(Y)$. We can always take K/k to be finite, and even finite separable if G is k-smooth (as then Y is k-smooth).

Remark B.1.1. The traditional language of Galois cohomology with smooth G as in [Se2, I, §5] uses the set $\mathrm{Z}^1(k_s/k, G)$ of continuous functions $g : \sigma \mapsto g_\sigma$ on $\mathrm{Gal}(k_s/k)$ valued in the discrete set $G(k_s)$ and satisfying the 1-cocycle relation $g_{\sigma\tau} = \sigma(g_\tau) \cdot g_\sigma$. Continuous functions $\mathrm{Gal}(k_s/k) \to G(k_s)$ are induced by functions $\mathrm{Gal}(k'/k) \to G(k')$ for sufficiently large finite Galois extensions k'/k inside of k_s, and so can be viewed as elements of $\prod_{\sigma \in \mathrm{Gal}(k'/k)} G(k') = G(\prod_\sigma k')$. By using the k-algebra isomorphism $\alpha : k' \otimes_k k' \simeq \prod_\sigma k'$ defined by $a' \otimes b' \mapsto (a'\sigma(b'))_\sigma$ and the k-algebra isomorphism $\alpha \circ (\alpha \otimes 1) : k'^{\otimes 3} \simeq \prod_{\sigma,\tau} k'$ defined by $a' \otimes b' \otimes c' \mapsto (a'\sigma(b')\tau(c'))$, we can identify these functions with elements of $G(k' \otimes_k k')$. Passing to the direct limit over k', we identify $\mathrm{Z}^1(k_s/k, G)$ with the set of elements $\gamma \in G(k_s \otimes_k k_s)$ such that $p_{13}^*(\gamma) = p_{23}^*(\gamma) \cdot p_{12}^*(\gamma)$ in $G(k_s \otimes_k k_s \otimes_k k_s)$. (Look at the $(\sigma, \sigma\tau)$-factor field of $k'^{\otimes 3}$.) This latter point of view, working with group functors on k-algebras and their values on the k-algebras K, $K \otimes_k K$, and $K \otimes_k K \otimes_k K$, likewise defines non-abelian degree-1 Čech theory for group functors relative to any field extension K/k.

For any k-group scheme G of finite type, we can translate the construction of ${}_YG$ into the language of non-abelian Čech theory relative to any field extension K/k that splits the G-torsor Y (and may be taken to be of finite degree), as follows. For any $y_0 \in Y(K)$ the points $p_1^*(y_0), p_2^*(y_0) \in Y(K \otimes_k K)$ satisfy $p_1^*(y_0) = p_2^*(y_0).\gamma_0$ for a unique $\gamma_0 \in G(K \otimes_k K)$. By applying pullback along the three canonical k-algebra maps $K \otimes_k K \to K \otimes_k K \otimes_k K$ we see that γ_0 is a 1-cocycle in the sense that $p_{13}^*(\gamma_0) = p_{23}^*(\gamma_0) \cdot p_{12}^*(\gamma_0)$ in $G(K \otimes_k K \otimes_k K)$. The set $\mathrm{H}^1(K/k, \underline{\mathrm{Aut}}(G))$ of k-isomorphism classes of k-groups that become isomorphic to G over K is described as follows in terms of non-abelian degree-1 Čech theory: it is the quotient of the set $\mathrm{Z}^1(K/k, \underline{\mathrm{Aut}}(G)) \subseteq \mathrm{Aut}_{K \otimes_k K}(G_{K \otimes_k K})$ of 1-cocycles of the functor $\underline{\mathrm{Aut}}(G)$ relative to K/k modulo the equivalence relation $\varphi \sim \varphi'$ defined by the property $\varphi' = p_2^*(\psi) \circ \varphi \circ p_1^*(\psi)^{-1}$ for some $\psi \in \mathrm{Aut}_K(G_K)$. The natural map of group functors $G \to \underline{\mathrm{Aut}}(G)$ carrying g to the conjugation operation $c_g : x \mapsto gxg^{-1}$ induces a map of pointed sets $\mathrm{H}^1(K/k, G) \to \mathrm{H}^1(K/k, \underline{\mathrm{Aut}}(G))$ whose image is (by definition) the set of *inner forms* of G split by K/k.

The k-group ${}_YG$ is an inner form of G because for any $y_0 \in Y(K)$ with associated $\gamma_0 \in G(K \otimes_k K)$ as above we have $[p_1^*(y_0), p_1^*(y_0).g] = [p_2^*(y_0), p_2^*(y_0).(\gamma_0 g \gamma_0^{-1})]$ in $({}_YG)(R)$ for any $K \otimes_k K$-algebra R and any $g \in G(R)$ (so the cohomology class of γ_0 in $\mathrm{H}^1(K/k, G)$ maps to the class of ${}_YG$ in $\mathrm{H}^1(K/k, \underline{\mathrm{Aut}}(G))$). If we replace y_0 with some $y_1 \in Y(K)$ then $y_1 = y_0.g$ for a unique $g \in G(K)$, so the associated 1-cocycle in $\mathrm{Z}^1(K/k, G) \subseteq G(K \otimes_k K)$ is $p_2^*(g) \cdot \gamma_0 \cdot p_1^*(g)^{-1}$; this is visibly cohomologous to γ_0.

B.2 Twisting of torsors

We can make Y into a right ${}_YG$-torsor over k by the requirement $y_1.[y_2, y_1] = y_2$. It is easy to check that this is indeed a well-defined right torsor structure over k, and it is denoted Y'. In terms of the isomorphism $\underline{\mathrm{Aut}}_G(Y) \simeq {}_YG$, this makes $\underline{\mathrm{Aut}}_G(Y)$ act on Y on the right through inversion in this group functor (with its usual left action on Y). We can repeat this construction to make an inner form ${}_{Y'}({}_YG)$ of ${}_YG$. There is a unique k-group isomorphism $\iota_G : G \simeq {}_{Y'}({}_YG)$ that sends any $g \in G(R)$ (for a k-algebra R) to the common class $[y, y.g^{-1}]$ for all points y of Y_R; the use of g^{-1} rather than g is needed to make ι_G a homomorphism of group schemes. An equivalent formulation is that G is naturally the functor of automorphisms of the underlying k-scheme Y that commute with the action of $\underline{\mathrm{Aut}}_G(Y)$ on Y; this is a "double centralizer" property for torsors.

The twisting operation $G \rightsquigarrow {}_YG \simeq \underline{\mathrm{Aut}}_G(Y)$ on the k-group G by the fixed right G-torsor Y can be generalized to convert right G-torsors into right ${}_YG$-torsors as follows. If X is any right G-torsor then define ${}_YX$ to be the quotient of $Y \times X$ modulo the diagonal right G-action $(y, x).g = (y.g, x.g)$. As for ${}_YG$ in §B.1, this is easily checked to be represented by a quasi-projective k-scheme, and if $X = G$ is the trivial right G-torsor then this recovers ${}_YG$ as defined above. There is no evident G-action on ${}_YX$ in general, and ${}_YX$ is really a torsor for another k-group, namely ${}_YG$. To describe this conceptually, we first note that by assigning to $(y, x) \in Y(R) \times X(R)$ the unique G_R-torsor isomorphism $Y_R \simeq X_R$ carrying y to x, we realize ${}_YX$ as representing the Isom-functor $\underline{\mathrm{Isom}}_G(Y, X)$ of G-torsor isomorphisms from Y to X. On $\underline{\mathrm{Isom}}_G(Y, X)$ there is an evident structure of right torsor over $\underline{\mathrm{Aut}}_G(Y) \simeq {}_YG$. More concretely, there is a unique well-defined right action of ${}_YG$ on ${}_YX$ over k determined by the rule $[y, x].[y, y.g] = [y, x.g]$, and this makes ${}_YX$ into a right ${}_YG$-torsor over k. It is clear that the k-isomorphism class of the right ${}_YG$-torsor ${}_YX$ only depends on the k-isomorphism class of the right G-torsor X, so at the level of sets of isomorphism classes over k we get a well-defined map of sets $t_{Y,k} : \mathrm{H}^1(k, G) \to \mathrm{H}^1(k, {}_YG)$ carrying $[X]$ to $[{}_YX]$.

We claim that the map $t_{Y,k}$ is bijective. This can be seen by arguing in terms of Isom and Aut functors as just described, but let us give another argument by providing a construction in the opposite direction and verifying in terms of the quotient constructions of ${}_YG$ and ${}_YX$ that it is an inverse. Since Y has been endowed with a structure of right ${}_YG$-torsor (which we denoted Y'), for any right ${}_YG$-torsor Z we get the right ${}_{Y'}({}_YG)$-torsor ${}_{Y'}Z$ and via the canonical isomorphism $\iota_G : G \simeq {}_{Y'}({}_YG)$ this is a right G-torsor. The reader can check that the map $X \to {}_{Y'}({}_YX)$ defined by carrying each $x \in X(R)$ (for a k-algebra R) to the common equivalence class of pairs $(y, [y, x])$ for points y of Y_R is a torsor isomorphism over k that is equivariant with respect to ι_G. Thus, $t_{Y,k} : \mathrm{H}^1(k, G) \to \mathrm{H}^1(k, {}_YG)$ defined by $[X] \mapsto [{}_YX]$ has a left inverse provided by the map $\mathrm{H}^1(k, {}_YG) \to \mathrm{H}^1(k, {}_{Y'}({}_YG)) \simeq \mathrm{H}^1(k, G)$ defined by $[Z] \mapsto [{}_{Y'}Z]$. If Z is a right ${}_YG$-torsor over k and the right ${}_{Y'}({}_YG)$-torsor ${}_{Y'}Z$ is viewed as a right G-torsor via ι_G then the twist ${}_Y({}_{Y'}Z)$ by the right G-torsor Y is a right ${}_YG$-torsor over k that is naturally isomorphic to Z. (Concretely, ${}_Y({}_{Y'}Z)$ is the quotient of $Y \times ({}_{Y'}Z)$ modulo the equivalence relation $(y_1, [y_1, z]) \sim (y_2, [y_2, z])$ for all $y_1, y_2 \in Y$ and $z \in Z$, and the resulting k-scheme isomorphism $Z \simeq {}_Y({}_{Y'}Z)$ carrying each z to the common equivalence class of $(y, [y, z])$ for all $y \in Y$ is equivariant with respect to the k-group isomorphism ${}_Y(\iota_G) : {}_YG \simeq {}_Y({}_{Y'}({}_YG))$.) We have therefore constructed an inverse to the twisting map in cohomology.

B.3 Exact sequences via torsors and gerbes

Let $1 \to G' \xrightarrow{j} G \xrightarrow{\pi} G'' \to 1$ be a short exact sequence of finite type k-group schemes. There is a naturally associated 6-term complex of pointed sets

$$1 \to G'(k) \to G(k) \to G''(k) \xrightarrow{\delta_0} \mathrm{H}^1(k, G') \to \mathrm{H}^1(k, G) \to \mathrm{H}^1(k, G'') \tag{B.3.1}$$

in which $\delta_0(g'')$ is the fiber $\pi^{-1}(g'')$ viewed as a right G'-torsor. (In the language of torsors, the composite map $G''(k) \to \mathrm{H}^1(k, G)$ carries $g'' \in G''(k)$ to the pushout of $\pi^{-1}(g'')$ along $j : G' \hookrightarrow G$, and that pushout is a trivial G-torsor since the canonical inclusion $\pi^{-1}(g'') \hookrightarrow G$ is equivariant with respect to the right actions of G' on the source and G on the target. Thus, (B.3.1) is indeed a complex.) In case G', G, and G'' are all smooth, this complex coincides with the habitual one in Galois cohomology as in [Se2, I, §5.7].

Our main goal in this section is to discuss the 7-term exact sequence of pointed sets obtained when G' is central in G, especially the interaction of the connecting map $\delta_1 : \mathrm{H}^1(k, G'') = \mathrm{H}^1(\overline{k}/k, G'') \to \mathrm{H}^2(k, G')$ with the twisting methods in §B.1–§B.2. The subtlety is that if G' is not smooth then Čech methods do *not* apply (because $G(\overline{k} \otimes_k \overline{k}) \to G''(\overline{k} \otimes_k \overline{k})$ is generally not surjective when G' is not smooth). The definition of δ_1 therefore requires going beyond Čech methods, relying on gerbes (which we review below).

Remark B.3.1. The only place we use δ_1 with non-smooth central G' and non-commutative G or G'' is in the proof of Proposition 7.1.3, which in turn is not used anywhere in this paper. Thus, the reader who is familiar with the Galois cohomological approach to δ_1 in the smooth case (especially the twisting aspect in [Se2, I, §5.7, Prop. 44]) and does not care about Proposition 7.1.3 may ignore the rest of this section.

Proposition B.3.2. *The complex of pointed sets* (B.3.1) *is exact.*

Proof. Only at two steps is this not a tautology: at $\mathrm{H}^1(k, G')$ and $\mathrm{H}^1(k, G)$. First consider a right G'-torsor Y', so there exists $y' \in Y'(K)$ for some finite-degree extension field K/k inside of $\overline{k}$. Then $p_1^*(y') = p_2^*(y').g'$ for $g' \in \mathrm{Z}^1(K/k, G') \subseteq G'(K \otimes_k K)$ that represents the class of Y' in $\mathrm{H}^1(K/k, G')$. If $\mathrm{H}^1(k, G') \to \mathrm{H}^1(k, G)$ kills the class of Y' then by replacing K with a suitable finite extension there exists $g \in G(K)$ such that $p_2^*(g) = p_1^*(g)j(g')$, so applying π yields the equality $p_2^*(\pi(g)) = p_1^*(\pi(g))$. Hence, by faithfully flat descent, the element $\pi(g) \in G''(K)$ comes from some $g'' \in G''(k)$. The right G'-torsor $\pi^{-1}(g'')$ over k splits over K by using the base point g, so $\delta_0(g'')$ is represented by the unique $g'_1 \in G'(K)$ such that $p_2^*(g) = p_1^*(g)j(g'_1)$. Clearly $g'_1 = g'$ by uniqueness, so $\delta_0(g'')$ is the class of Y'.

Next consider a right G-torsor Y over k that is split by pushout along $\pi : G \to G'$. We want to show that Y is the pushout of a G'-torsor along j. Let K/k be a finite extension such that Y_K is split, so for a choice of $y_0 \in Y(K)$ we have $p_2^*(y_0) = p_1^*(y_0)g$ for some $g \in \mathrm{Z}^1(K/k, G) \subseteq G(K \otimes_k K)$. By hypothesis, $\pi(g) = p_1^*(g'')^{-1}p_2^*(g'')$ for some $g'' \in G''(K)$. Increasing K by a finite amount, we have $g'' = \pi(g_1)$ for some $g_1 \in G(K)$. Replacing y_0 with $y_0.g_1$ (as we may) then brings us to the case that $\pi(g) = 1$, so $g = j(g')$ for some $g' \in G'(K \otimes_k K)$. Since j is an inclusion, g' inherits the 1-cocycle condition from g and so defines a right G'-torsor Y' whose pushout along j is Y. ■

In the special case that G' is central in G (i.e., G'-conjugation on G is trivial), there is a derived functor cohomology group $\mathrm{H}^2(k, G')$; the natural map $\check{\mathrm{H}}^2(k, G') \to \mathrm{H}^2(k, G')$ is *injective*, due to the limiting form of the Čech-to-derived functor cohomology spectral sequence. The centrality of G' in G ensures that if G' is *smooth* then the habitual Čech-theoretic definition of the connecting map of pointed sets $\delta : \mathrm{H}^1(k, G'') \to \check{\mathrm{H}}^2(k, G')$ makes sense (i.e., $G(\overline{k} \otimes_k \overline{k}) \to G''(\overline{k} \otimes_k \overline{k})$ is surjective when G' is smooth). Unfortunately, there is no such connecting map to Čech cohomology when G' is not smooth, so we need another method to allow for such G': the interpretation of commutative H^2 (not just $\check{\mathrm{H}}^2$) in terms of gerbes.

We now review what a gerbe is. Let S be a scheme (such as $\operatorname{Spec} k$) and A an abelian sheaf for the fppf topology on the category of locally finitely presented S-schemes (e.g., the sheaf represented by an fppf S-group scheme). An *A-gerbe* on S is a stack fibered in groupoids $\mathscr{X} \to S$ for the fppf topology on the category of locally finitely presented S-schemes such that $\mathscr{X}(T)$ is non-empty for some covering $T \to S$ and for any $S' \to S$ and $\xi \in \mathscr{X}(S')$ the following two conditions hold: (i) the automorphism group of ξ is identified with $A(S')$ functorially in ξ and S', and (ii) for any $\eta \in \mathscr{X}(S')$, ξ and η are isomorphic locally on S'. For example, the fibered category $\mathrm{Tor}_{A/S}$ of A-torsors (in sheaves of sets) is an A-gerbe (the *trivial* A-gerbe), and it is characterized (up to isomorphism) by the condition that $\mathscr{X}(S) \neq \emptyset$: for any $\xi \in \mathscr{X}(S)$, the functor $\mathscr{X} \to \mathrm{Tor}_{A/S}$ defined on S'-fibers by $\eta \mapsto \underline{\mathrm{Isom}}(\xi_{S'}, \eta)$ is an isomorphism (with inverse defined by the effective descent condition in $\mathscr{X}$). Thus, loosely speaking, an A-gerbe over S is a kind of twisted descent of $\mathrm{Tor}_{A_{S'}/S'}$ relative to a covering map $S' \to S$ for the fppf topology. In particular, it is easy to define a *pullback* functor on A-gerbes relative to any map of schemes $T \to S$. (If A is an fppf S-group scheme then A-gerbes are Artin stacks and so can be studied geometrically, but we do not use this deep fact since we need to allow general abelian group sheaves A.)

The pointed set $\mathrm{H}^2_{\mathrm{g}}(S, A)$ of isomorphism classes of A-gerbes over S makes sense and is functorial in S via pullback. By the universal δ-functor arguments with general abelian group sheaves in [Mi1, §2.5, Ch. IV] (and references therein), the pointed set $\mathrm{H}^2_{\mathrm{g}}(S, A)$ has a natural functoriality in A and as such is identified with the abelian group $\mathrm{H}^2(S, A)$. Moreover, for any short exact sequence $0 \to A' \to A \to A'' \to 0$ in fppf abelian group sheaves over S, the connecting map

$$\delta : \check{\mathrm{H}}^1(S, A'') = \mathrm{H}^1(S, A'') \to \mathrm{H}^2(S, A') = \mathrm{H}^2_{\mathrm{g}}(S, A') \tag{B.3.2}$$

is explicitly described as follows: it associates to the isomorphism class of any A''-torsor P'' the isomorphism class of the A-gerbe $\delta(P'')$ whose fiber over any locally finitely presented S-scheme T is the groupoid of pairs (P, α) where P is an A-torsor over T and $\alpha : P \times^A A'' \simeq P''_T$ is an isomorphism of A''-torsors over T (and an *isomorphism* $(P_1, \alpha_1) \simeq (P_2, \alpha_2)$ is defined to be an isomorphism of A-torsors $P_1 \simeq P_2$ over T carrying α_1 to α_2). Note that $\delta(P'')$ is an A'-gerbe because $A \to A''$ is surjective with central kernel A'.

The explicit description of (B.3.2) makes sense when A and A'' are not commutative, provided that A' is central in A (and we use right torsors, for specificity). This motivates the following useful result.

PROPOSITION B.3.3. *Let $1 \to Z \to G \to G'' \to 1$ be a short exact sequence of finite type group schemes over a field k, with Z central in G. Define the connecting map of pointed sets $\delta_1 : \mathrm{H}^1(k, G'') \to \mathrm{H}^2(k, Z)$ by the procedure with gerbes as in the description of* (B.3.2).

(i) *The formation of δ_1 is functorial in k and the diagram $\mathrm{H}^1(k, G) \to \mathrm{H}^1(k, G'') \xrightarrow{\delta_1} \mathrm{H}^2(k, Z)$ is an exact sequence of pointed sets.*

(ii) *Let Y'' be a right G''-torsor. The connecting map $\mathrm{H}^1(k, {}_{Y''}G'') \to \mathrm{H}^2(k, Z)$ associated to the Y''-twisted central extension $1 \to Z \to {}_{Y''}G \to {}_{Y''}G'' \to 1$ is carried to δ_1 via composition with the natural bijection of sets $t_{Y'',k} : \mathrm{H}^1(k, G'') \simeq \mathrm{H}^1(k, {}_{Y''}G'')$.*

In (ii), ${}_{Y''}G$ denotes the k-form of G defined via $\mathrm{H}^1(k, G'') \to \mathrm{H}^1(\overline{k}/k, \underline{\mathrm{Aut}}(G))$ using the "conjugation" action of G'' on its central extension G; it is generally *not* an inner form of G.

Proof. The exactness in (i) is a tautology via the definition of δ_1 and the meaning of the triviality of a gerbe. The explicit description of the Y''-twisting operation in §B.2 via ${}_{Y''}X'' = \underline{\mathrm{Isom}}_{G''}(Y'', X'')$ and ${}_{Y''}G'' = \underline{\mathrm{Aut}}_{G''}(Y'')$ makes it easy to verify (ii) by hand. ∎

Appendix C. Proof of Proposition 3.2.1 for smooth groups

In §3.2 we reduced the proof of the general case of Proposition 3.2.1 to the smooth case via Lemma 3.1.1. In this section we give a "modern" proof of the result in the smooth case (bypassing the crutch of GL_n), so now use notation as in Proposition 3.2.1 and assume G is smooth. Pick a finite non-empty set S of places of k (containing the archimedean places) for which there exists a map of $\mathscr{O}_{k,S}$-groups $G^0_S \to G_S$ as considered above the statement of Proposition 3.2.1. Since G is k-smooth, we may and do arrange (by enlarging S if necessary) that G_S is smooth over $\mathscr{O}_{k,S}$. For S' containing S, let $\mathbf{A}_{k,S'} = (\prod_{v\in S'} k_v) \times \prod_{v\notin S'} \mathscr{O}_v$ as a topological product ring (open in $\mathbf{A}_k$) and let $G_{S'} = G_S \otimes_{\mathscr{O}_{k,S}} \mathscr{O}_{k,S'}$ and $G^0_{S'} = G^0_S \otimes_{\mathscr{O}_{k,S}} \mathscr{O}_{k,S'}$.

Clearly $G^0_{S'}(\mathbf{A}_{k,S'})$ is a closed subgroup of $G_{S'}(\mathbf{A}_{k,S'})$: this is the inclusion

$$\prod_{v\in S'} G^0(k_v) \times \prod_{v\notin S'} G^0_S(\mathscr{O}_v) \to \prod_{v\in S'} G(k_v) \times \prod_{v\notin S'} G_S(\mathscr{O}_v).$$

The inclusions in each factor are open and closed embeddings. Clearly $G(k_v)/G^0(k_v)$ injects into $(G/G^0)(k_v)$, which is a finite set (since G/G^0 is finite étale over k). Likewise, for $v \notin S$ we see that $G_S(\mathscr{O}_v)/G^0_S(\mathscr{O}_v)$ injects into the finite set $G(k_v)/G^0(k_v)$ (since $G^0_S(\mathscr{O}_v) = G^0(k_v) \cap G_S(\mathscr{O}_v)$ inside of $G(k_v)$, due to G^0_S being closed in G_S). Thus, $G_S(\mathbf{A}_{k,S})/G^0_S(\mathbf{A}_{k,S})$ is topologically a product of finite discrete groups, so it is profinite.

For finite S' containing S, since $\mathbf{A}_{k,S'} = \mathscr{O}_{k,S'} \otimes_{\mathscr{O}_{k,S}} \mathbf{A}_{k,S}$ and G^0_S is closed in G_S we see that

$$G^0_S(\mathbf{A}_{k,S}) = G_S(\mathbf{A}_{k,S}) \cap G^0_{S'}(\mathbf{A}_{k,S'})$$

inside of $G_{S'}(\mathbf{A}_{k,S'}) = G_S(\mathbf{A}_{k,S'})$. Thus, the continuous map of profinite groups

$$f_{S',S} : G_S(\mathbf{A}_{k,S})/G^0_S(\mathbf{A}_{k,S}) \to G_{S'}(\mathbf{A}_{k,S'})/G^0_{S'}(\mathbf{A}_{k,S'})$$

is injective and hence is a closed embedding. However, $G_S(\mathbf{A}_{k,S})$ is open in $G_{S'}(\mathbf{A}_{k,S'})$, so the closed embedding $f_{S',S}$ between profinite groups is also an open embedding, whence it has finite index. The same holds with (S, S') replaced by (Σ, Σ') for any finite sets Σ and Σ' of places of k containing S with $\Sigma \subseteq \Sigma'$.

Since $G(\mathbf{A}_k)$ is the directed union of open subgroups $G_{S'}(\mathbf{A}_{k,S'})$, and similarly for G^0 with the groups $G^0_{S'}$, $G(\mathbf{A}_k)/G^0(\mathbf{A}_k)$ is the directed union of open subgroups $G_{S'}(\mathbf{A}_{k,S'})/G^0_{S'}(\mathbf{A}_{k,S'})$ with their profinite quotient topologies. Thus, our problem is exactly to prove that this directed chain stops. It is equivalent to show that for all sufficiently large S', $G_{S'}(\mathbf{A}_{k,S'})G^0(k_v)$ contains $G(k_v)$ for every $v \notin S'$. That is, $G_S(\mathscr{O}_v)G^0(k_v) \stackrel{?}{=} G(k_v)$ for all but finitely many v (outside S).

Here is the key point: if we consider the short exact sequence

$$1 \to G^0 \stackrel{j}{\to} G \stackrel{\pi}{\to} G/G^0 \to 1$$

of k-group schemes, the map π is smooth, separated, and faithfully flat (i.e., surjective), with G/G^0 a finite (étale) k-group scheme, so by standard "spreading out" arguments we can enlarge

S such that there is a finite (étale) $\mathscr{O}_{k,S}$-group scheme E_S with generic fiber G/G^0 and a smooth, separated, surjective $\mathscr{O}_{k,S}$-group scheme map $\pi_S : G_S \to E_S$ of finite type that recovers π over k. The kernel $H_S := \ker(\pi_S)$ is a smooth separated finite type $\mathscr{O}_{k,S}$-group scheme, and its generic fiber is identified with G^0. Thus, by increasing S we may find an isomorphism $H_S \simeq G^0_S$ compatible with the closed immersions into G_S. By increasing S we can therefore "spread out" the maps j and π to maps in an exact sequence

$$1 \to G^0_S \stackrel{j_S}{\to} G_S \stackrel{\pi_S}{\to} E_S \to 1$$

of finite type separated $\mathscr{O}_{k,S}$-group schemes. (That is, j_S is a closed immersion that identifies G^0_S with the kernel of the faithfully flat π_S.)

Now it suffices to prove the following well-known claim. Suppose R is a complete (or just henselian) discrete valuation ring with fraction field K and *finite* residue field κ, and G is a separated smooth finite type group scheme over R. Suppose there is a smooth surjection $G \to E$ onto a finite R-group scheme E such that the (necessarily smooth) kernel $\mathscr{G}$ has (necessarily geometrically) connected fibers, so $\mathscr{G}_x = G^0_x$ for each $x \in \operatorname{Spec}(R)$. Then we claim that $G(K) = \mathscr{G}(K)G(R)$. Since $E(K) = E(R)$ by finiteness of E, and $G(K) \to E(K)$ has kernel $\mathscr{G}(K)$, we just have to show that for any $g \in G(K)$, its image $\overline{g} \in E(K) = E(R)$ is in the image of $G(R)$. In other words, we want $X(R) \neq \emptyset$, where X is the pullback in the cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & G \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} R & \xrightarrow[\overline{g}]{} & E \end{array}$$

Since $G \to E$ is a torsor for the smooth E-group $E \times_{\operatorname{Spec} R} \mathscr{G}$, X is a torsor for the R-group $\mathscr{G}$. Lang's theorem ensures that torsors for smooth connected groups over finite fields are trivial, so $X(\kappa)$ is non-empty. But X is R-smooth and R is henselian local, so a rational point on the closed fiber lifts to an R-point.

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